Extending the reach of the point-to-set principle

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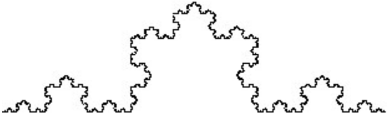
Fractal dimensions

Given a (separable) metric space, Hausdorff dimension and packing dimension generalize the usual integer dimension idea



Hausdorff definition of dimension

Let ρ be a metric on a set X.

- For $E \subseteq X$ and $\delta > 0$, a δ-cover of E is a collection $\mathcal{U}$ such that for all $U \in \mathcal{U}$, $\mathrm{diam}(U) < \delta$ and
$$E \subseteq \bigcup_{U \in \mathcal{U}} U.$$
- For $s \geq 0$,
$$H^s(E) = \lim_{\delta \to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s$$

The Hausdorff dimension of $E \subseteq X$ is
$$\dim_{\mathrm{H}}(E) = \inf \{ s \,|\, H^s(E) = 0 \} .$$

Gauged dimension

We can avoid infinite dimension by changing the scale /gauge function families

- A **gauge function** is a continuous, nondecreasing function from $[0, \infty)$ to $[0, \infty)$ that vanishes only at 0.

- A **gauge family** is a one-parameter family $\varphi = \{\varphi_s \,|\, s \in (0, \infty)\}$ of gauge functions φ_s satisfying for $s > t$, $\varphi_s(\delta) = o(\varphi_t(\delta))$ as $\delta \to 0^+$

Definition

$$H^{s,\varphi}(E) = \lim_{\delta \to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U \in \mathcal{U}} \varphi_s(\mathrm{diam}(U))$$

$$\mathrm{dim}^{\varphi}(E) = \inf \{ s \,|\, H^{s,\varphi}(E) = 0 \} .$$

They generalize $\theta_s(\delta) = \delta^s$ in Hausdorff dimension.

Packing dimension

Let ρ be a metric on a set X.

- For $E \subseteq X$ and $\delta > 0$, a δ-packing of E is a collection $\mathcal{U}$ of disjoint open balls U with centers in E and $\mathrm{diam}(U) < \delta$.
- For $s \geq 0$,
  $$P_0^s(E) = \lim_{\delta \to 0} \sup_{\mathcal{U} \text{ is a } \delta\text{-packing of } E} \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s$$
- For $s \geq 0$,
  $$P^s(E) = \inf \{\textstyle\sum_i P_0^s(E_i) \,|\, E \subseteq \cup E_i\}$$

The Packing dimension of $E \subseteq X$ is
$$\dim_{\mathrm{P}}(E) = \inf \{s \,|\, P^s(E) = 0\}.$$

Can also be generalized using gauge functions

Algorithmic dimensions on 2^ω

- Effectivizing the (several equivalent) definitions of fractal dimension we obtain algorithmic dimensions on Cantor and Euclidean spaces
- We can effectivize in several ways:
  - From a martingale/gale definition of dimension: an s-gale is $d : 2^{<\omega} \to [0, \infty)$ with

$$d(w) = \frac{d(w0) + d(w1)}{2^s},$$

$$S^\infty[d] = \left\{ x \in 2^\omega \,\middle|\, \limsup_n (x \restriction n) = \infty \right\}.$$

$$S^\infty_{\text{strong}}[d] = \left\{ x \in 2^\omega \,\middle|\, \liminf_n (x \restriction n) = \infty \right\}.$$

  - From a compression/decompression definition: Fix U a UTM. Let $w \in 2^{<\omega}$, $x \in 2^\omega$, $\delta > 0$

$$\mathrm{K}(w) = \min \{ |y| \,|\, U(y) = w \}$$

$$\mathrm{K}_\delta(x) = \inf \{ \mathrm{K}(q) \,|\, q \in \mathbb{Q}, |x - q| < \delta \}$$

Effective dimension

Definition

Let $x \in 2^\omega$ and $A \subseteq 2^\omega$

$$\dim(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)}.$$

$$\mathrm{Dim}(x) = \limsup_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)}.$$

They are equivalent to:

$$\dim(x) = \inf\{s \mid \text{there is a lower semicomputable } s\text{-gale } d \text{ with } x \in S^\infty[d]\}$$

(an similarly for Dim and S_{strong}).

All definitions relativize to any oracle $B \subseteq \mathbb{N}$.

Effective dimension

These effectivizations are **pointwise**

- $\dim(A) = \sup_{x \in A} \dim(x)$
- $\mathrm{Dim}(A) = \sup_{x \in A} \mathrm{Dim}(x)$

Effective dimension

Definition

The φ-*gauged algorithmic dimension* of a point $x \in 2^\omega$ is

$$\dim^\varphi(x) = \inf \left\{ s \,\middle|\, \liminf_{\delta \to 0^+} 2^{\mathrm{K}_\delta(x)} \varphi_s(\delta) = 0 \right\}.$$

What does it have to do with fractal geometry?

$\dim_{\mathrm{H}}(A)$ is the **Hausdorff dimension** of set A
$\dim_{\mathrm{P}}(A)$ is the **packing dimension** of set A

For $A \subseteq 2^{\omega}$

$$\dim_{\mathrm{H}}(A) \leq \dim(A)$$

$$\dim_{\mathrm{P}}(A) \leq \mathrm{Dim}(A)$$

What does it have to do with fractal geometry?

$$\dim_{\mathrm{H}}(A) \leq \dim(A)$$

$$\dim_{\mathrm{P}}(A) \leq \mathrm{Dim}(A)$$

- Useful for upper bounds
- For sets of low complexity, $\dim_{\mathrm{H}}(A) = \dim(A)$ and $\dim_{\mathrm{P}}(A) \leq \mathrm{Dim}(A)$ (correspondence principles)
- Partial randomness of points, quantitative analysis of sets ...

Point-to-set principles

Theorem (Lutz Lutz 2018)

Let $A \subseteq 2^\omega$. Then

$$\dim_{\mathrm{H}}(A) = \min_{B \subseteq \mathbb{N}} \dim^B(A).$$

Theorem (Lutz Lutz 2018)

Let $A \subseteq 2^\omega$. Then

$$\dim_{\mathrm{P}}(A) = \min_{B \subseteq \mathbb{N}} \mathrm{Dim}^B(A).$$

Application of point to set principles to fractal geometry: projection formula

Theorem (Marstrand 1954)

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_{\mathrm{H}}(E) = s$. Then for almost every $\theta \in (0, 2\Pi)$, $\dim_{\mathrm{H}}(p_\theta E) = \min\{s, 1\}$

It does not hold for arbitrary E (assuming CH). Recently an extension using PSP

Theorem (N.Lutz Stull 2018)

Let $E \subseteq \mathbb{R}^2$ be an arbitrary set with $\dim_{\mathrm{H}}(E) = \dim_{\mathrm{P}}(E) = s$. Then for almost every $\theta \in (0, 2\Pi)$, $\dim_{\mathrm{H}}(p_\theta E) = \min\{s, 1\}$

Further extension in (Stull 2021)

Intersection formula

Theorem (Kahane 1986, Mattila 1984)

Let $E, F \subseteq \mathbb{R}^n$ *be* **Borel sets**. *Then for almost every* $z \in \mathbb{R}^n$,

$$\dim_{\mathrm{H}}(E \cap (F + z)) \leq \max\{0, \dim_{\mathrm{H}}(E \times F) - n\}$$

where $F + z = \{x + z \,|x \in F\}$.

Theorem (N.Lutz 2021)

Let $E, F \subseteq \mathbb{R}^n$. *Then for almost every* $z \in \mathbb{R}^n$,

$$\dim_{\mathrm{H}}(E \cap (F + z)) \leq \max\{0, \dim_{\mathrm{H}}(E \times F) - n\}$$

where $F + z = \{x + z \,|x \in F\}$.

Other

- (Lutz Stull 2020) results on Furstenberg sets
- (Slaman 2021) The Hausdorff dimensions of co-analytic sets are not carried by their closed subsets
- (Lutz 2021) There are Hamel bases ($\mathbb{R}$ over $\mathbb{Q}$) with any positive Hausdorff dimension

Looking at other separable spaces

- Where can we effectivize dimension?
  - We can define Kolmogorov complexity/ effectivize Hausdorff measure if we have a **separator** (countable dense set)

Definition (Kolmogorov complexity of x at precision δ)

Let (X, ρ) be a separable metric space and let $D \subseteq X$ be a countable dense set (fix $f : 2^{<\omega} \twoheadrightarrow D$)

$$\mathrm{K}_\delta(x) = \inf \left\{ \mathrm{K}(w) \,\middle|\, w \in 2^{<\omega}, \rho(x, f(w)) < \delta \right\}$$

Looking at other separable spaces

Definition

The *algorithmic dimension* and strong algorithmic dimension of a point $x \in X$ is

$$\dim(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)},$$

$$\mathrm{Dim}(x) = \limsup_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)}.$$

Looking at other spaces: gauge dimension

Definition

The *φ-gauged algorithmic dimension* and *s*trong algorithmic dimension of a point $x \in X$ is

$$\dim^{\varphi}(x) = \inf \left\{ s \,\middle|\, \liminf_{\delta \to 0^+} 2^{\mathrm{K}_\delta(x)} \varphi_s(\delta) = 0 \right\},$$

and the *φ-gauged* of x is

$$\mathrm{Dim}^{\varphi}(x) = \inf \left\{ s \,\middle|\, \limsup_{\delta \to 0^+} 2^{\mathrm{K}_\delta(x)} \varphi_s(\delta) = 0 \right\},$$

Ordinary Hausdorff/packing dimensions use the *canonical gauge family* defined by $\theta_s(\delta) = \delta^s$

A couple of restrictions on gauge families

$\varphi = \{\varphi_s \mid s \in (0, \infty)\}$ is a **strong gauge family** if the following hold:

- There is a precision familiy: a one-parameter family $\alpha = \{\alpha_s \mid s \in (0, \infty)\}$ of functions $\alpha_s : \mathbb{N} \to \mathbb{Q}^+$ that vanish as $r \to \infty$ and satisfy
  - $\varphi_s(\alpha_s(r)) = O(\varphi_s(\alpha_s(r+1)))$ as $r \to \infty$
  - $\sum_{r \in \mathbb{N}} \frac{\varphi_t(\alpha_s(r))}{\varphi_s(\alpha_s(r))} < \infty$ whenever $s < t$.
- $\varphi_t(2\delta) = O(\varphi_s(\delta))$ for all $s \leq t$
- $\varphi_s(\delta) = O(1/\log\log(1/\delta))$ as $\delta \to 0^+$

General Point-to-set principles

Let (X, ρ) be a separable metric space, φ a strong gauge family

Theorem (Lutz Lutz M 2021)

Let $A \subseteq X$. Then

$$\dim_{\mathrm{H}}^{\varphi}(A) = \min_{B \subseteq \mathbb{N}} \sup_{x \in A} \dim^{\varphi, B}(x).$$

Theorem (Lutz Lutz M 2021)

Let $A \subseteq X$. Then

$$\dim_{\mathrm{P}}^{\varphi}(A) = \min_{B \subseteq \mathbb{N}} \sup_{x \in A} \mathrm{Dim}^{\varphi, B}(x).$$

The hyperspace

- Let (X, ρ) be a separable metric space
- Let $\mathcal{K}(X)$ be the set of nonempty compact subsets of X together with the Hausdorff metric $\mathrm{dist_H}$ defined as follows

$$\mathrm{dist_H}(U, V) = \max\left\{\sup_{x \in U} \rho(x, V), \sup_{y \in V} \rho(y, U)\right\}.$$

$(\rho(a, B) = \inf \{\rho(a, b) \,|\, b \in B\})$

Relationship of the dimensions of E and $\mathcal{K}(E)$

McClure (1995 and 1996) has several results relating Hausdorff and packing dimensions of a set E and $\mathcal{K}(E)$ for

- E self-similar
- E σ-compact

$\mathcal{K}(E)$ has infinite dimension, a different gauge family is needed

The jump of a gauge family

Definition

The *jump* of a gauge family φ is the family $\widetilde{\varphi}$ given $\widetilde{\varphi}_s(\delta) = 2^{-1/\varphi_s(\delta)}$.

For the canonical gauge family $\theta_s(\delta) = \delta^s$, $\widetilde{\theta}_s(t) = 2^{-1/t^s}$.

Theorem (McClure 1995)

*Let $E \subseteq X$ be σ-**compact**. Let $\psi_s(t) = 2^{-1/t^s}$. Then*

$$\dim_{\mathrm{P}}^{\psi}(\mathcal{K}(E)) \geq \dim_{\mathrm{P}}(E).$$

We aim to extend the theorem to other E and to other gauge families beside the canonical one.

Hyperspace packing dimension theorem

Theorem (LLM 2021)

Let $E \subseteq X$ be an analytic set, and let φ be strong gauge family, then

$$\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) \geq \dim_{\mathrm{P}}^{\varphi}(E).$$

Where we use PSP in the hyperspace packing dimension theorem

- By the general point-to-set principle, let A be an oracle such that
$$\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) = \sup_{L \in \mathcal{K}(E)} \mathrm{Dim}^{\widetilde{\varphi},A}(L),$$
- We recursively construct a single compact set $L \in \mathcal{K}(E)$ (i.e., a single point in the hyperspace $\mathcal{K}(E)$) that has high Kolmogorov complexity at infinitely many precisions, relative to oracle A.
$$\mathrm{Dim}^{\widetilde{\varphi},A}(L) > s$$
- For E compact, we can reach $\mathrm{Dim}^{\widetilde{\varphi},A}(L) > s$ for $s = \dim_{\mathrm{P}}^{\varphi}(E)$

Open questions

- Is there a more general hyperspace Hausdorff dimension theorem? $\dim_{\mathrm{H}}^{\widetilde{\varphi}}(\mathcal{K}(E))$ vs $\dim_{\mathrm{H}}^{\varphi}(E)$ for interesting E
- Are (the complexity or the good properties of) the two oracles in the PTSP related to hyperspace dimension theorems?

References

- Jack H. Lutz and Neil Lutz, Who asked us? How the theory of computing answers questions about analysis, Ding-Zhu Du and Jie Wang (eds.), Complexity and Approximation: In Memory of Ker-I Ko, pp. 48-56, Springer, 2020.
- Jack H. Lutz and Elvira Mayordomo, Algorithmic fractal dimensions in geometric measure theory. In Vasco Brattka and Peter Hertling (eds.), Handbook of Computability and Complexity in Analysis, Springer-Verlag (2021).
- Jack H. Lutz, Neil Lutz, and Elvira Mayordomo, Extending the Reach of the Point-to-Set Principle. `https://arxiv.org/abs/2004.07798`