

COALGEBRAS AND CORECURSIVE ALGEBRAS IN CONTINUOUS MATHEMATICS

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Logicians are very familiar with **recursion** and **induction**, perhaps less so with their “duals” **corecursion** and **coinduction**.

Many of the fundamental structures in mathematical logic happen to be **initial algebras**:
the **natural numbers**, or the **cumulative hierarchy of sets**.

At the same time, there are many compelling structures that are characterized as **final coalgebras**:
the Cantor space, **the unit interval**, **fractals**, and **Harsanyi type spaces**.

This talk is a high-level introduction to the area of **coalgebra**, tuned to a logic audience.

Part of the appeal of **coalgebra** in theoretical computer science is that it gives a set of tools relevant and applicable to **finitely approximable infinite objects**.

These same tools can be pointed back at more “classical” topics,
like those in areas of continuous mathematics.

This talk is a kind of progress report on this turn.

It is more like an examination of special topics
and less of a general theory.

Let \mathcal{A} be a category, and let $F : \mathcal{A} \rightarrow \mathcal{A}$ be a functor.

An **F -algebra** is a morphism of the form $a : FA \rightarrow A$.

An **initial algebra** is one with a unique morphism to any algebra.

$$\begin{array}{ccc}
 FA & \xrightarrow{a} & A \\
 F\varphi \downarrow & & \downarrow \text{there is a unique morphism } \varphi \\
 FB & \xrightarrow{\quad} & B \\
 & \text{for all } b &
 \end{array}$$

The category is **Set**.

The functor is $FX = 1 + X$.

An algebra for F is a set A together with a map

$$1 + A \rightarrow A$$

So it is an element $a \in A$ and an endo-map $f : A \rightarrow A$.

The main example is $N = \omega$, the natural numbers, with $0 \in N$, and $s : N \rightarrow N$ the successor function.

A morphism of algebras is what you think it is.

RECURSION ON N IS TANTAMOUNT TO INITIALITY

Recursion on N : For all sets A , all $a \in A$, and all $f : A \rightarrow A$, there is a unique $\varphi : N \rightarrow A$ so that

$$\begin{aligned}\varphi(0) &= a \\ \varphi(n+1) &= f(\varphi(n)) \quad \text{for all } n\end{aligned}$$

Initiality of N : For all $(A, [a, f])$, there is a unique homomorphism

$$\varphi : (N, [0, s]) \rightarrow (A, [a, f])$$

That is, the diagram below commutes:

$$\begin{array}{ccc} 1 + N & \xrightarrow{[0, s]} & N \\ \downarrow 1 + \varphi & & \downarrow \varphi \\ 1 + A & \xrightarrow{[a, f]} & A \end{array}$$

Recursion on N may be recast in terms of maps out of an initial algebra.

The category is **Set**.

The functor is $FX = 1 + (X \times X)$.

An algebra for F is a set A together with a map

$$1 + (A \times A) \rightarrow A$$

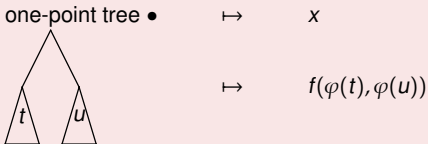
So it is an element of A and a map $a : A \times A \rightarrow A$.

A morphism of algebras is what you think it is.

EXAMPLE: $FX = 1 + (X \times X)$

RECURSION PRINCIPLE FOR FINITE TREES

For all sets X , all $x \in X$, all $f : X \times X \rightarrow X$,
there is a unique $\varphi : \mu F \rightarrow X$
so that φ is



RECURSION PRINCIPLE FOR FINITE TREES

For all **algebras** $f : 1 + (X \times X) \rightarrow X$,
there is a unique $\varphi : \mu F \rightarrow X$ so that

$$\begin{array}{ccc}
 F(\mu F) & \xrightarrow{?} & \mu F \\
 \downarrow 1+(\varphi \times \varphi) & & \downarrow \varphi \\
 FX & \xrightarrow{f} & X
 \end{array}$$

commutes, where $(\varphi \times \varphi)(t, u) = (\varphi(t), \varphi(u))$.

Let \mathcal{A} be a category, and let $F : \mathcal{A} \rightarrow \mathcal{A}$ be a functor.

An **F -algebra** is a morphism of the form $a : FA \rightarrow A$.

An **F -coalgebra** is a morphism of the form $a : A \rightarrow FA$.

Example: deterministic automata

$$(S, s : S \rightarrow 2 \times S^A)$$

are coalgebras of $2 \times X^A$, again on Set.

MORPHISMS OF ALGEBRAS AND COALGEBRAS

Let $(A, a : FA \rightarrow A)$ and $(B, b : FB \rightarrow B)$ be algebras.

A **morphism** is $f : A \rightarrow B$ in the same underlying category so that

$$\begin{array}{ccc} FA & \xrightarrow{a} & A \\ Ff \downarrow & & \downarrow f \\ FB & \xrightarrow{b} & B \end{array}$$

commutes.

Let $(A, a : A \rightarrow FA)$ and $(B, b : B \rightarrow FB)$ be coalgebras.

A **morphism** is $f : A \rightarrow B$ in the same underlying category so that

$$\begin{array}{ccc} A & \xrightarrow{a} & FA \\ f \downarrow & & \downarrow Ff \\ B & \xrightarrow{b} & FB \end{array}$$

commutes.

INITIAL ALGEBRAS AND FINAL COALGEBRAS

initial algebra

$$\begin{array}{ccc} FA & \xrightarrow{a} & A \\ Ff \downarrow & & \downarrow f \\ FB & \xrightarrow{b} & B \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{a} & FA \\ f \downarrow & & \downarrow Ff \\ B & \xrightarrow{b} & FB \end{array}$$

final coalgebra

functor	initial algebra
$1 + X$ on <i>Set</i>	natural numbers
$1 + X^2$ on <i>Set</i>	finite binary trees
$1 + (A \times X)$ on <i>Set</i>	finite sequences from A
$1 + X^2$ on <i>MS</i>	finite binary trees, with metric
$1 + X^2$ on <i>CMS</i>	finite and infinite binary trees, with metric

functor	final coalgebra
$1 + X$ on <i>Set</i>	natural numbers + ∞
$1 + X^2$ on <i>Set</i>	finite and infinite binary trees
$1 + (A \times X)$ on <i>Set</i>	finite and infinite sequences from A

In all these cases, the **structure maps** are also natural.

EXAMPLE: FORMAL LANGUAGES AS A FINAL COALGEBRA

The category is Set .

The functor is $F(S) = 2 \times S^A$, where A is a fixed “alphabet” set.

Coalgebras of $2 \times X^A$ are deterministic automata

$$\begin{array}{ccc} S & \xrightarrow{s} & 2 \times S^A \\ \varphi \downarrow & & \downarrow id_2 \times \varphi^A \\ \mathcal{L} & \xrightarrow{I} & 2 \times \mathcal{L}^A \end{array} \quad \text{final coalgebra}$$

Let $\mathcal{L} = \mathcal{P}(A^*)$ be the set of **formal languages over A**

The final coalgebra is

$$\mathcal{L} \rightarrow 2 \times \mathcal{L}^A,$$

and is given in terms of Brzozowski derivatives.

The map φ takes a state to the language accepted there.

The theme is that a lot of interesting structures in mathematics, starting with the set of natural numbers itself, are either initial algebras or final coalgebras.

What about the set \mathbb{R} of reals?

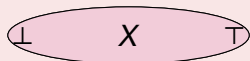
What about $[0, 1]$?

Let BiP be the category of **bi-pointed sets**.

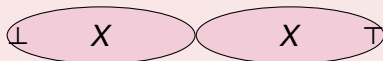
These are triples (X, \top, \perp) with X a set and also $\top, \perp \in X$ and $\top \neq \perp$.

The bipointed set $\{\top, \perp\}$ is initial, but there is no final object.

THE FUNCTOR $F : \text{BiP} \rightarrow \text{BiP}$



\mapsto



identify \top of left with \perp of right

THE MAIN EXISTENCE THEOREM FOR INITIAL ALGEBRAS/FINAL COALGEBRAS

ADÁMEK 1974

Assume that the underlying category \mathcal{A} has an initial object 0 and a **colimit** μF of

$$0 \xrightarrow{!} F0 \xrightarrow{F1} F^2 0 \xrightarrow{F^2 1} \dots \quad \xrightarrow{F^{n-1} 1} F^n 0 \xrightarrow{F^n 1} \dots$$

and that the functor $F : \mathcal{A} \rightarrow \mathcal{A}$ preserves this ω -colimit.

There is a canonical morphism $m : F(\mu F) \rightarrow F$ such that

$$(\mu F, m)$$

is an **initial** F -algebra.

EXAMPLE

Take \mathcal{A} to be the category of bipointed sets, and F as above.

0 is $\{\top, \perp\}$.

The category has all colimits, and F preserves the colimit of the chain above.

Indeed, that colimit is \mathcal{D} the dyadic rationals in $[0, 1]$.

So the initial algebra is $F\mathcal{D} \rightarrow \mathcal{D}$.

THE MAIN EXISTENCE THEOREM FOR INITIAL ALGEBRAS/FINAL COALGEBRAS

BARR 1993

Assume that the underlying category \mathcal{A} has a final object 1 and a **limit** νF of

$$1 \xleftarrow{!} F1 \xleftarrow{F!} F^2 1 \xleftarrow{F^2!} \dots \xleftarrow{F^{n-1}!} F^n 1 \xleftarrow{F^n!} \dots$$

and that $F : \mathcal{A} \rightarrow \mathcal{A}$ preserves this ω -limit.

There is a canonical morphism $m : F \rightarrow F(\nu F)$ such that

$$(\nu F, m)$$

is a **final** F -coalgebra.

We **cannot** use this to prove Freyd's characterization of $[0, 1]$.

Recall

$$i : [0, 1] \rightarrow F[0, 1].$$

These sets are **complete metric spaces**,
 i is a bijection and an isometry.

Regard the set X a (discrete) space.

The space

$$S = \text{hom}_{\text{BiP}}(X, [0, 1]).$$

is a closed subspace of $\text{hom}_{\text{CMS}}(X, [0, 1])$, hence is complete.

$$d(Ff, Fg) \leq \frac{1}{2}d(f, g).$$

We have a **contracting** endofunction $\psi : S \rightarrow S$:
 take $f : X \rightarrow [0, 1]$ to $i^{-1} \cdot Ff \cdot e$:

$$\begin{array}{ccc}
 X & & X \xrightarrow{e} FX \\
 f \downarrow & & \downarrow \psi(f) \quad \downarrow Ff \\
 [0, 1] & & [0, 1] \xleftarrow{i^{-1}} F[0, 1]
 \end{array}$$

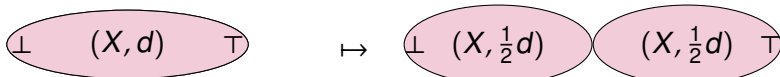
By the Contraction Mapping Thm., there's a unique $f^* = \psi(f^*)$.

f^* is exactly a coalgebra morphism $(X, e) \rightarrow ([0, 1], i)$.

EXERCISE

I'm cheating. But how?

Consider $F : \text{BiMS} \rightarrow \text{BiMS}$ given by



identify \top of left with \perp of right
use the quotient metric

FREYD

The final coalgebra of $F : \text{BiP} \rightarrow \text{BiP}$ is $[0, 1]$ as above.

VARIATION: LM

The final coalgebra of $F : \text{BiMS} \rightarrow \text{BiMS}$ is $[0, 1]$ as above.

- ▶ On BiP, the initial algebra is the dyadic rationals in $[0, 1]$.
- ▶ On BiP, the final coalgebra is the unit interval as a set.
- ▶ On BiMS, the final coalgebra is the unit interval as a metric space.

The final coalgebra turned out to be the Cauchy completion of the initial algebra.

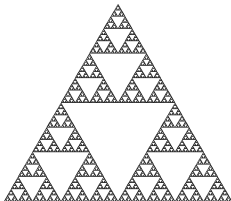
To some extent, fractal subsets of \mathbb{R}^n are described as final coalgebras.

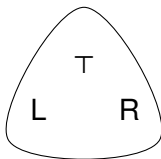
In many cases those final coalgebras are completions of the initial algebras.

This has been worked out in a few concrete settings:

- ▶ the Sierpinski triangle and the circle(!)
(with Prasit Bhattacharya, Jayampathy Ratnayake, and Robert Rose)
- ▶ the Sierpinski gasket, including complex gluing.
(with Victoria Noquez)

THE SIERPIŃSKI GASKET AS A FINAL COALGEBRA





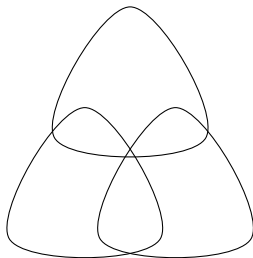
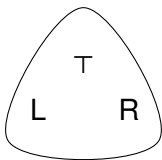
A **tripointed set** is a set X together with distinguished different elements \top , L , and R .

Morphisms are functions preserving \top , L , and R .

The initial object I of Tri is $\{\top, L, R\}$.

But Tri has no final object.

Here is a generic tripointed set:



The functor F takes this to 3 copies with identifications as shown above. In a tripointed metric space:

- ▶ all 3 distinguished points have distance 1
- ▶ the functor squashes distances by $1/2$

category	initial algebra	final coalgebra
Set ₃ Tripointed sets	(G, g) “finite address space” of the gasket \mathcal{S}	its completion (S, s) also $(\mathcal{S}, \sigma) =$ the Sierpinski Gasket as a subset of \mathbb{R}^2
Met ₃ ^{Sh} short maps	(G, g)	(S, s)
Met ₃ ^L Lipschitz maps	(G_ρ, g) G with discrete metric	none exists
Met ₃ ^C continuous maps	(G_ρ, g)	(S, s) and (\mathcal{S}, σ) they are bilipschitz isomorphic

The presentation of $[0, 1]$ and \mathbb{R} is “defective” in the sense that it doesn’t relate to the natural arithmetic operations that we use on those structures.

To capture these, we need another definition.

$(B, \beta : FB \rightarrow B)$ is a **corecursive algebra for F**

if for every coalgebra $\alpha : A \rightarrow FA$, there is a unique $\alpha^\dagger : A \rightarrow B$ so that

$$\alpha^\dagger = \beta \cdot F\alpha^\dagger \cdot \alpha$$

as shown:

$$\begin{array}{ccc}
 A & \xrightarrow{\alpha} & FA \\
 \downarrow \alpha^\dagger & & \downarrow F\alpha^\dagger \\
 B & \xleftarrow{\beta} & FB
 \end{array}$$

corecursive algebra: $(\forall \alpha)(\exists! \alpha^\dagger)$

FOR $FX = N \times X$

$$N \times [0, 1] \xrightarrow{(n,r) \mapsto \frac{n+r}{1+n+r}} [0, 1]$$

$$N \times \mathbb{R}^{\geq 0} \xrightarrow{(n,r) \mapsto n + \frac{r}{1+r}} \mathbb{R}^{\geq 0}$$

Given $e : X \rightarrow N \times X$, we want a unique e^\dagger :

$$\begin{array}{ccc}
 X & \xrightarrow{e} & N \times X \\
 \downarrow f & & \downarrow N \times f \\
 [0, 1] & \xleftarrow{s(n,r) = \frac{n+r}{1+n+r}} & N \times [0, 1]
 \end{array}$$

We consider $\varphi : [0, 1]^X \rightarrow [0, 1]^X$ given by

$$\varphi(f) = s \cdot (N \times f) \cdot e$$

The space $[0, 1]^X$ is compact by **Tychonoff's Theorem**.

φ is a **shrinking map**: for $f \neq g$, $d(\varphi f, \varphi g) < d(f, g)$.

A shrinking map on a non-empty compact space has **a unique fixed point** (exercise!)

Let's adopt notation for $f : X \rightarrow \mathbb{R}^{\geq 0} \times X$:

$$f(x_0) = (a_0, x_1) \quad f(x_1) = (a_1, x_2) \quad \cdots \quad f(x_n) = (a_n, x_{n+1}) \quad \cdots$$

Then we are asking if we can solve the system

$$\begin{aligned} x_0 &= a_0 + \frac{1}{1+x_1} \\ x_1 &= a_1 + \frac{1}{1+x_2} \\ &\vdots \\ x_n &= a_n + \frac{1}{1+x_{n+1}} \\ &\vdots \end{aligned}$$

$$x_0 = a_0 + 1/(1 + a_1 + 1/(1 + a_2 + \cdots + 1/(1 + a_n + \cdots)))$$

The theory of **contined fractions** implies that we have a corecursive algebra.

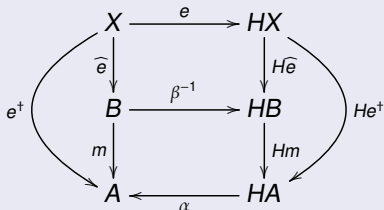
EXTRACTING FINAL COALGEBRAS FROM CERTAIN CORECURSIVE ALGEBRAS

LEMMA (ADÁMEK, MILIUS, LM)

Let H be any endofunctor, let (A, α) be a corecursive H -algebra.

Let (B, β) be a fixed point of H which is a subalgebra of A .

Assume that for every coalgebra $e : X \rightarrow HX$, the coalgebra-to-algebra map e^+ factors through the algebra morphism $m : B \rightarrow A$.

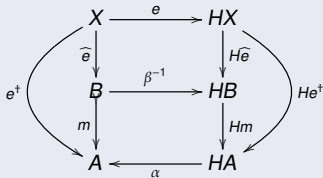


Then (B, β^{-1}) is the final coalgebra of H .

EXTRACTING FINAL COALGEBRAS FROM CERTAIN CORECURSIVE ALGEBRAS

PROOF.

Fix a coalgebra $e : X \rightarrow HX$. Consider the diagram below:



We claim that \widehat{e} is a coalgebra morphism.
For this, a diagram chase shows that

$$m \cdot \widehat{e} = m \cdot \beta \cdot H\widehat{e} \cdot e$$

Then use the fact that m is monic.

Uniqueness is also easy.

□

APPLICATION: THE POSITIVE IRRATIONALS

$$\begin{array}{ccc} X & \xrightarrow{e} & N \times X \\ e^+ \downarrow & & \downarrow N \times e^+ \\ [0, 1] & \xleftarrow{s(n,r) = \frac{n+r}{1+n+r}} & N \times [0, 1] \end{array}$$

LEMMA

For all coalgebras $e : X \rightarrow FX$ and all $x \in X$, $e^+(x)$ is an irrational number.

PROOF.

By contradiction, as in the proof that $\sqrt{2}$ is irrational, and also using a point in the justification of the Euclidean algorithm for gcd. □

APPLICATION: THE POSITIVE IRRATIONALS

Thus we have

$$\begin{array}{ccc}
 X & \xrightarrow{e} & N \times X \\
 \widehat{e} \downarrow & & \downarrow N \times \widehat{e} \\
 [0, 1] \cap Irr & \xrightarrow{\beta^{-1}} & N \times ([0, 1] \cap Irr) \\
 m \downarrow & & \downarrow N \times m \\
 [0, 1] & \xleftarrow{s(n,r) = \frac{n+r}{1+n+r}} & N \times [0, 1]
 \end{array}$$

e^+ (curved arrow from X to $[0, 1]$) $N \times e^+$ (curved arrow from $N \times X$ to $N \times [0, 1]$)

Here m is an inclusion,
 and the restriction of s to $[0, 1] \cap Irr$ is a bijection.

Thus, this restriction is a final coalgebra.

LEMMA

Let H be any endofunctor, let (A, α) be a corecursive H -algebra.

Let (B, β) be a fixed point of H which is a subalgebra of A .

Assume that for every coalgebra $e : X \rightarrow HX$, the coalgebra-to-algebra map e^+ factors through the algebra morphism $m : B \rightarrow A$.

Then (B, β^{-1}) is the final coalgebra of H .

Take A to be $(\mathbb{R}^{\geq 0}, \rho)$.

Take B to be the subalgebra of **positive irrationals**.

Let \mathcal{B} be the irrationals in $[0, 1]$.

Then

$$\mathcal{B} \cong B = \text{positive irrationals}$$

via

$$x \mapsto \frac{1}{x}.$$

We already know that (B, ρ^{-1}) is a final coalgebra for $N \times X$, where ρ is

$$n, r \mapsto n + \frac{1}{1+r}$$

Then ρ^{-1} transfers along the isomorphism.

PRATT & PAVLOVIC'S THEOREM

The Baire space

$$\mathcal{B} = [0, 1] \cap Irr$$

is a final coalgebra of $N \times X$.

The structure is

$$\langle \beta, \gamma \rangle : \mathcal{B} \rightarrow N \times \mathcal{B}$$

where

$$\beta(x) = \left\lfloor \frac{1}{x} \right\rfloor - 1 \quad \text{and} \quad \gamma(x) = \left(\frac{1}{x} \right) \bmod 1$$

γ is called the Gauss map.

ALL KNOWN FINAL COALGEBRAS
WHOSE CARRIERS ARE SUBSETS OF \mathbb{R}
AND WHOSE STRUCTURES ARE “SIMPLE” FUNCTIONS

$$FX = N \times X$$

carrier	inverse of structure
$\mathbb{R}^{\geq 0}$	$n + \frac{r}{1+r}$
$[0, 1)$	$\frac{n+r}{1+n+r}$
$\mathbb{R}^{\geq 0} \cap Irr$	$n + \frac{1}{1+r}$
$[0, 1] \cap Irr$	$\frac{1}{1+n+r}$

$$GX = N \times X + 1$$

carrier	inverse of structure
$\mathbb{R}^{\geq 0}$	$[\rho, 0]$, where $\rho(n, r) = n + \frac{1}{1+r}$
$[0, 1]$	$[\sigma, 1]$, where $\sigma(n, r) = \frac{n+r}{1+n+r}$

IS THIS A CORECURSIVE ALGEBRA?

The functor is

$$HX = \mathbb{R} \times X$$

And the algebra is

$$\mathbb{R} \times \mathbb{R} \xrightarrow{x+\gamma \cdot y} \mathbb{R}$$

with a fixed $0 < \gamma < 1$.

So we are asking whether every f has a unique f^\dagger :

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{R} \times X \\ f^\dagger \downarrow & & \downarrow \mathbb{R} \times f^\dagger \\ \mathbb{R} & \xleftarrow{x+\gamma \cdot y} & \mathbb{R} \times \mathbb{R} \end{array}$$

IS THIS A CORECURSIVE ALGEBRA?

Let's adopt notation for $f : X \rightarrow \mathbb{R} \times X$ like

$$f(x_0) = (a_0, x_1) \quad f(x_1) = (a_1, x_2) \quad \cdots \quad f(x_n) = (a_n, x_{n+1}) \quad \cdots$$

Then we are asking if we can solve the system

$$\begin{aligned}x_0 &= a_0 + \gamma x_1 \\x_1 &= a_1 + \gamma x_2 \\&\vdots \\x_n &= a_n + \gamma x_{n+1} \\&\vdots\end{aligned}$$

$$x_0 = a_0 + \gamma a_1 + \gamma^2 a_2 + \cdots + \gamma^n a_n + \cdots$$

IS THIS A CORECURSIVE ALGEBRA?

ALMOST

The functor is

$$HX = \mathbb{R} \times X$$

And the algebra is

$$\mathbb{R} \times \mathbb{R} \xrightarrow{x+\gamma \cdot y} \mathbb{R}$$

with a fixed $0 < \gamma < 1$.

LEMMA

For every *bounded* f there is a unique *bounded* f^\dagger :

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{R} \times X \\ f^\dagger \downarrow & & \downarrow \mathbb{R} \times f^\dagger \\ \mathbb{R} & \xleftarrow{x+\gamma \cdot y} & \mathbb{R} \times \mathbb{R} \end{array}$$

Bounded here means that $\sup_{x \in X} \pi_1(f(x)) < \infty$.

DEFINITION

An **ordered metric space** is $\mathcal{M} = (M, d)$ such that

- ▶ d is a metric on M
- ▶ \leq is a partial order on the set M ,
- ▶ for all $y \in M$,

$$\{z : z \leq y\} \text{ and } \{z : z \geq y\}$$

are closed sets in the metric topology.

The space \mathcal{M} is **complete** if it is complete as a metric space.

EXAMPLE

For all sets X , $B(X, \mathbb{R})$ is a complete ordered metric space if we consider it with the pointwise order, i.e.:

$$f \leq g \text{ iff } f(x) \leq g(x) \text{ for all } x \in X.$$

CONTRACTION (Co)INDUCTION PRINCIPLE

THEOREM (FEYS, HANSEN, LM'19)

Let \mathcal{M} be a complete ordered metric space.

Let $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ be **both** contracting and order preserving.

Then the fixed point Φ^* of Φ is also

- ▶ a least pre-fixed-point (if $\Phi(x) \leq x$, then $\Phi^* \leq x$),
- ▶ and a greatest post-fixed-point (if $x \leq \Phi(x)$, then $x \leq \Phi^*$).

PROOF.

Let x be such that $\Phi(x) \leq x$.

Then by induction on $n \geq 1$, $\Phi^n(x) \leq x$.

And since $\Phi^n(x) \leq x$ for all n , and $\{z : z \leq x\}$ is closed, we see that $\Phi^* = \lim_n \Phi^n(x) \leq x$ also. □

Now suppose that we have **two** contracting and order preserving operators, say $\Phi, \Psi : \mathcal{M} \rightarrow \mathcal{M}$.

LEMMA (SUFFICIENT CONDITION LEMMA)

If $\Phi(\Psi^) \leq \Psi^*$, then $\Phi^* \leq \Psi^*$.*

If $\Phi^ \leq \Psi(\Phi^*)$, then $\Phi^* \leq \Psi^*$.*

LEMMA

If P is a stochastic $n \times n$ matrix
(its columns are probability vectors), then

$$(I - \gamma P)^{-1}$$

has all non-negative entries.

USING CONTRACTION COINDUCTION

Let $\Phi : M \rightarrow M$ be

$$\Phi(X) = I + (\gamma P)X.$$

Easily, Φ is a monotone contraction,
and its fixed point is

$$\Phi^* = (I - \gamma P)^{-1}.$$

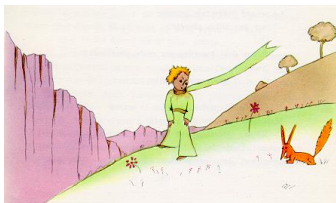
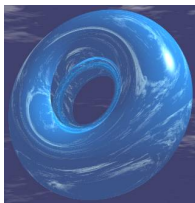
Let 0 be the zero matrix, and note that $\Phi(0) \geq 0$.

So 0 is a post-fixed-point.

By Contraction Coinduction, $(I - \gamma P)^{-1} = \Phi^* \geq 0$.

THE LITTLE PRINCE ON A TOROIDIAL PLANET

The Little Prince from
Antoine de Saint-Exupéry's book
lands on a **torus**,
a donut shaped planet
divided into nine regions (states).



He has a map of his planet.

It shows the **immediate reward** for each state.

-1 <i>a</i>	-1 <i>b</i>	great food 10 <i>c</i>
-1 <i>d</i>	sand -5 <i>e</i>	monster -4 <i>f</i>
avg food 5 <i>g</i>	-1 <i>h</i>	-1 <i>i</i>

At each step, the Little Prince has a choice to go N, S, E, or W. He cannot stay stationary.

Since the planet is a torus (donut), going N in state *a* leads to *g*, going W leads to *c*, etc.

His set of possible actions is

$$Act = \{\uparrow, \Rightarrow, \downarrow, \Leftarrow\}.$$

However, these actions are probabilistic.

If he chooses \uparrow , then with prob .8 he goes N , with prob .1 he actually lands E , and with prob .1 he actually lands W .

If he chooses \Rightarrow , then with prob .8 he goes E , with prob .1 he actually lands N , and with prob .1 he actually lands S .

If he chooses \downarrow , then with prob .8 he goes S , with prob .1 he actually lands E , and with prob .1 he actually lands W .

If he chooses \Leftarrow , with prob .8 he goes W , with prob .1 he actually lands N , and with prob .1 he actually lands S .

MARKOV DECISION PROCESSES: THE FORMAL DEFINITION

An **MDP** is a tuple $(S, Act, go, reward, \gamma)$ such that

x

- ▶ S is a **state set** whose elements are $s_1 \dots, s_n$
- ▶ Act is a set of **actions**, whose elements are α, β, \dots
- ▶ $go(s, \alpha, t)$, is a **probability transition function**, and for each s and α , we require that $\sum_t go(s, \alpha, t) = 1$.
- ▶ $reward : S \rightarrow \mathbb{R}$ is an **immediate reward function**;
- ▶ γ is a number between 0 and 1 called the **discount factor**.

We have not yet discussed γ .

The **distribution monad** is the discrete variant of the **Giry monad**.

The functor part $\Delta: \text{Set} \rightarrow \text{Set}$ gives the finite probability measures.

The unit $\delta: id \Rightarrow \Delta$ gives the Dirac distribution, and the multiplication

$$\mu: \Delta\Delta \Rightarrow \Delta$$

is “mixing”.

Expected value is an Eilenberg-Moore algebra

$$E: \Delta\mathbb{R} \rightarrow \mathbb{R}$$

POLICIES AND THEIR LONG TERM VALUES

A **policy** is a function from states to actions:

$$\sigma: S \rightarrow \text{Act}$$

Each policy has a **long term value function**

$$\text{LTV}_\sigma: S \rightarrow \mathbb{R}$$

It is the unique bounded function making the following diagram commute:

$$\begin{array}{ccccc} S & \xrightarrow{m_\sigma} & \mathbb{R} \times \Delta S & & \\ \text{LTV}_\sigma \downarrow & & \downarrow \mathbb{R} \times \Delta(\text{LTV}_\sigma) & & \\ \mathbb{R} & \xleftarrow{x + \gamma \cdot y} & \mathbb{R} \times \mathbb{R} & \xleftarrow{\mathbb{R} \times E} & \mathbb{R} \times \Delta \mathbb{R} \end{array}$$

I continue to omit all machinery handling the boundedness requirement.

$$\begin{array}{ccc}
 S & \xrightarrow{m_\sigma} & \mathbb{R} \times \Delta S \\
 \text{LTV}_\sigma \downarrow & & \downarrow \mathbb{R} \times \Delta(\text{LTV}_\sigma) \\
 \mathbb{R} & \xleftarrow{x+\gamma \cdot y} \mathbb{R} \times \mathbb{R} & \xleftarrow{\mathbb{R} \times E} \mathbb{R} \times \Delta \mathbb{R}
 \end{array}$$

The map on the bottom is a corecursive algebra.

In some of this work, we need a metric on $\Delta \mathbb{R}$, and we take the **Kantorovich metric**:

$$d_{\Delta X}(\varphi, \psi) = \sup\{d_{\mathbb{R}}((E \circ \Delta f)(\varphi), (E \circ \Delta f)(\psi)) \mid f: X \rightarrow \mathbb{R} \text{ is non-expansive}\}$$

We got

$$\begin{array}{ccc}
 S & \xrightarrow{m_\sigma} & \mathbb{R} \times \Delta S \\
 \text{LTV}_\sigma \downarrow & & \downarrow \mathbb{R} \times \Delta(\text{LTV}_\sigma) \\
 \mathbb{R} & \xleftarrow{x+\gamma \cdot y} \mathbb{R} \times \mathbb{R} \xleftarrow{\mathbb{R} \times E} & \mathbb{R} \times \Delta \mathbb{R}
 \end{array}$$

from a **determinized version**

$$\begin{array}{ccc}
 \Delta S & \xrightarrow{\det m_\sigma} & \mathbb{R} \times \Delta S \\
 \ell'_\sigma \downarrow & & \downarrow \mathbb{R} \times \ell'_\sigma \\
 \mathbb{R} & \xleftarrow{x+\gamma \cdot y} & \mathbb{R} \times \mathbb{R}
 \end{array}$$

that was obtained via a **distributive law**

Bellman asks for a map $V^* : S \rightarrow \mathbb{R}$ so that

$$V^*(s) = u(s) + \gamma \cdot \max_{a \in \text{Act}} \left\{ \sum_{s' \in S} t_a(s)(s') \cdot V^*(s') \right\}.$$

We write this as

$$\begin{array}{ccc}
 S & \xrightarrow{m=(u,t)} & \mathbb{R} \times (\Delta S)^{\text{Act}} \\
 \downarrow v^* & & \downarrow \mathbb{R} \times (\Delta V^*)^{\text{Act}} \\
 \mathbb{R} & \xleftarrow{x + \gamma \cdot y \cdot (\mathbb{R} \times \max_{\text{Act}} \cdot E^{\text{Act}})} & \mathbb{R} \times (\Delta \mathbb{R})^{\text{Act}}
 \end{array}$$

The presence of “max” here makes this difficult to solve

A policy τ is called an **improvement** of a policy σ if for all $s \in S$ it holds that

$$\tau(s) = \operatorname{argmax}_{a \in Act} \{\ell_{\sigma}(t_a(s))\}.$$

Informally, $\tau(s)$ is an action a that maximizes the expected future rewards obtained by doing a now, and then continuing with σ .

However, it is **not prima facie clear** that τ is an improvement, since following τ means to also “continue with τ ” (not with σ).

Proving that $\sigma \leq \tau$ is the content of the Policy Improvement Theorem.

THEOREM (BELLMAN)

The Policy Improvement Theorem holds.

PROOF.

New proof [FHM] uses Contraction Coinduction. □

The importance of this on an ideological level is that Policy Improvement feels like coinduction!

A SURPRISE: EMAIL FROM OCTOBER 7

Hello,

My name is XXXXX and I am a PhD student at YYY.
I was recently on an internship at IBM Research where
I worked on formalizing the foundations of MDPs.

You might be interested in the paper we wrote, where
we directly formalized (in the proof assistant Coq)
Sections 2 and 3 of your paper "Long Term Values of
Markov Decision Processes (Co)algebraically".
<https://arxiv.org/abs/2009.11403>

I want to thank you for such a well-written paper, it
was a pleasure to read it and formalize proofs using
contraction coinduction!

Thank you,
XXXXX

I hope to have shown that there are ways to apply
final coalgebras and **corecursive algebras**
in settings related to classical continuous mathematics.

Much has yet to be done to get a systematic account.

THE CONCEPTUAL COMPARISON CHART

set with algebraic operations	set with transitions and observations
algebra for a functor	coalgebra for a functor
initial algebra	final coalgebra
least fixed point	greatest fixed point
congruence relation	bisimulation equivalence rel'n
equational logic	modal logic
recursion: map out of an initial algebra	corecursion: map into a final coalgebra
Foundation Axiom	Anti-Foundation Axiom
iterative conception of set	coiterative conception of set
useful in syntax	useful in semantics
useful in discrete math	useful in continuous math

MAIN REFERENCES, FROM MY OWN WORK

Jiří Adámek, Stefan Milius, and LM,
Initial Algebras, Terminal Coalgebras, and the
Theory of Fixed Points of Functors
(a draft of this book is on the web)

Prasit Bhattacharya, Jay. Ratnayake, Robert Rose, and LM,
Fractal Sets as Final Coalgebras Obtained by Completing an
Initial Algebra,
in *Horizons of the Mind. A Tribute to Prakash Panangaden*.

Frank Feys, Helle Hansen, and LM,
Long-term Values in Markov Decision Processes
(Co)Algebraically, CMCS'18.

Victoria Noquez and LM,
The Sierpinski Carpet as a Final Coalgebra, ms. 2021.