The classification problem for extensions of countable torsion abelian groups

Martino Lupini

Logic Seminar
University of Wisconsin, Madison
November 30, 2021

Table of Contents

Table of Contents

Extensions of groups

All the groups in this talk will be abelian.

Suppose that C, A are countable groups.

An extension of C by A is an exact sequence

$$0 \longrightarrow A \xrightarrow{\quad f \quad} X \xrightarrow{\quad p \quad} C \longrightarrow 0$$

Isomorphism of extensions

Consider two extensions of C by A:

$$0 \longrightarrow A \longrightarrow X \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow X' \longrightarrow C \longrightarrow 0$$

Sum of extensions

Consider two extensions of C by A:

$$0 \longrightarrow A \xrightarrow{f} X \xrightarrow{p} C \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{f'} X' \xrightarrow{p'} C \longrightarrow 0$$

The group Ext

$\mathrm{Ext}(C, A)$ is the set of isomorphism classes of extensions

$\mathrm{Ext}(C, A)$ is an abelian group

The operation on Ext is induced by sum of extension

Split extensions

The trivial element of $\mathrm{Ext}(C, A)$ is the class of split extensions

$$0 \longrightarrow A \xrightarrow{\quad f \quad} X \xrightarrow{\quad p \quad} C \longrightarrow 0$$

Extensions and cocycles

Consider an extension

$$0 \longrightarrow A \xrightarrow{\ f\ } X \underset{t}{\overset{p}{\rightleftarrows}} C \longrightarrow 0$$

where $p \circ t = \mathrm{id}_C$

Define then

$$\xi : C \times C \longrightarrow A,\ (x, y) \mapsto t(x+y) - t(x) - t(y)$$

This is a cocycle on C with values in A, i.e. it satisfies

- $\xi(x, y) = \xi(y, x)$
- $\xi(x+y, z) + \xi(x, y) = \xi(y, z) + \xi(x, y+z)$

Sum of extensions corresponds to pointwise sum of cocycles

Split extensions and coboundaries

We have that the extension is split if and only ξ is a coboundary

This means that there exists

$$r : C \longrightarrow A$$

such that

$$\xi(x, y) = r(x + y) - r(x) - r(y)$$

Another description of Ext

$\mathrm{Z}(C, A)$ is the group of cocycles on C with values in A

$\mathrm{B}(C, A) \subseteq \mathrm{Z}(C, A)$ is the subgroup of coboundaries

Then we have that

$$\mathrm{Ext}(C, A) \cong \mathrm{Z}(C, A)/\mathrm{B}(C, A)$$

The problem of classifying extensions up to isomorphism corresponds to the coset equivalence relation of $\mathrm{B}(C, A)$ in $\mathrm{Z}(C, A)$

The complexity of coboundaries

We have that $\mathrm{B}(C, A)$ is in general not a closed subgroup of $\mathrm{Z}(C, A)$

However $\mathrm{B}(C, A)$ is Borel, as it is the image of the continuous group homomorphism between Polish groups

$$\begin{array}{ccc} A^C & \longrightarrow & \mathrm{Z}(C, A) \\ r & \mapsto & ((x, y) \mapsto r(x + y) - r(x) - r(y)) \end{array}$$

Definability of coboundaries

This does not gives us an explicit way to express $\mathrm{B}(C, A)$ in terms of open sets by taking intersections and unions

In fact, it is not possible to give such a description (uniformly):

Theorem (L., 2021)

The subgroup $\mathrm{B}(C, A)$ is Borel in $\mathrm{Z}(C, A)$ but not uniformly in C, A

The Borel rank of coboundaries

To prove this, one can consider the Borel rank of $\mathrm{B}(C, A)$ in $\mathrm{Z}(C, A)$

This is the least $\alpha < \omega_1$ such that $\mathrm{B}(C, A) \in \mathbf{\Pi}^0_\alpha(\mathrm{Z}(C, A))$

Theorem (L., 2021)

The Borel rank of $\mathrm{B}(C, A)$ in $\mathrm{Z}(C, A)$ can be arbitrarily high for countable torsion groups C, A

Borel classes

One can refine the analysis by studying the complexity class of $\mathrm{B}(C, A)$

Definition

$\mathbf{\Pi}^0_\alpha$ is the complexity class of B in X if:

- B is $\mathbf{\Pi}^0_\alpha$ in X
- the complement of B is not $\mathbf{\Pi}^0_\alpha$ in X

Same definition for $\mathbf{\Sigma}^0_\alpha$ and and for $D\left(\mathbf{\Pi}^0_\alpha\right)$ (differences of $\mathbf{\Pi}^0_\alpha$ sets)

Possible complexity classes

Theorem (Hjorth–Kechris–Louveau, 1998)

A general result imposing restrictions on the possible complexity classes of $\mathrm{B}(C, A)$ *in* $\mathrm{Z}(C, A)$. *It must be one of the following:*

- $\mathbf{\Pi}^0_{1+\lambda+n}$ *where* λ *is either zero or limit and* $n = 0$ *or* $2 \leq n < \omega$*;*
- $\mathbf{\Sigma}^0_{1+\lambda+1}$ *where* λ *is either zero or limit;*
- $D\left(\mathbf{\Pi}^0_{1+\lambda+n}\right)$ *where* λ *is either zero or limit and* $2 \leq n < \omega$.

Complexity classes of cocycles

Theorem

Complete characterization of the complexity class of $\mathrm{B}(C, A)$ *in* $\mathrm{Z}(C, A)$ *in terms of the Ulm invariants of* A, C*, where* A, C *are torsion groups.*

All possible complexity classes are realized by suitable choices of C, A*.*

Example

Fix a prime p. Suppose that:

- $C = \mathbb{Z}(p^\infty)$ is the divisible p-group of p-rank 1;
- A is a countable unbounded reduced p-group.

The complexity class of $\mathrm{B}(C, A)$ in $\mathrm{Z}(C, A)$ is $\mathbf{\Pi}^0_{\alpha+2}$, where α is the least countable ordinal such that the α-th Ulm subgroup of A is bounded.

Table of Contents

Groups with a Polish cover

The group

$$\mathrm{Ext}(C, A) = \mathrm{Z}(C, A)/\mathrm{B}(C, A)$$

is an example of group with a Polish cover

In general a group with a Polish cover is an exact sequence

$$0 \longrightarrow K \xrightarrow{\varphi} \hat{G} \longrightarrow G \longrightarrow 0$$

where:

- K and $\hat{G}$ are Polish groups
- φ is a continuous group homomorphism

Morphisms

A morphism between groups with a Polish cover $G = \hat{G}/N$ and $H = \hat{H}/M$ is a group homomorphism $\varphi : G \to H$ that is Borel-definable

Borel-definable homological algebra

Theorem (L., 2021)

Groups with a Polish cover form an abelian category, which is an abelian subcategory of the category of groups

Polishable subgroups

Definition

If $G = \hat{G}/N$ is a group with a Polish cover, then a subgroup of G is Polishable if it is of the form

$$H = \hat{H}/N$$

where $\hat{H}$ is a Polishable subgroup of $\hat{G}$ containing N

The complexity class and Borel rank of H in G are by definition the complexity class and Borel rank of $\hat{H}$ in $\hat{G}$

Example

The complexity class of $\{0\}$ in $\mathrm{Ext}(C, A)$ is the complexity class of $\mathrm{B}(C, A)$ in $\mathrm{Z}(C, A)$

The first Solecki subgroup

Evidently, $\overline{N}/N = \overline{\{0\}}$ is the smallest closed subgroup of $G = \hat{G}/N$

Theorem (Solecki, 1999)

Let G be a group with a Polish cover.
There exists a smallest $\mathbf{\Pi}^0_3$ Polishable subgroup of G, denoted by $s_1(G)$

The Solecki subgroups

Let G be a group with a Polish cover.
One define recursively for $\alpha < \omega_1$, the Polishable subgroups:

- $s_0(G) = \overline{\{0\}}$;
- $s_{\alpha+1}(G) = s_1(s_\alpha(G))$;
- $s_\lambda(G) = \bigcap_{\beta<\lambda} s_\beta(G)$ for λ limit.

The Solecki rank $\rho(G)$ of G is the least $\alpha < \omega_1$ such that $s_\alpha(G) = \{0\}$.

The complexity of Solecki subgroups

Theorem (L., 2021)

Let G be a group with a Polish cover.
For $\alpha \leq \rho(G)$, $s_\alpha(G)$ is the smallest $\mathbf{\Pi}^0_{1+\alpha+1}$ Polishable subgroup of G.

- *If α is limit, then the Borel rank of $s_\alpha(G)$ is α;*
- *α is a successor, then the Borel rank of $s_\alpha(G)$ is $1+\alpha+1$.*

Moreover, one can express the complexity class of $s_\alpha(G)$ in G in terms of α and the complexity class of $s_\alpha(G)$ inside $s_{\alpha-1}(G)$ (if α is a successor).

Corollary

- *If $\rho(G)$ is limit, then the Borel rank of $\{0\}$ in G is $\rho(G)$.*
- *If $\rho(G)$ is a successor, then the Borel rank of $\{0\}$ in G is $1+\rho(G)+1$.*

Table of Contents

A particular case

We consider the case when

- $C = \mathbb{Z}(p^\infty)$ is the divisible p-group of p-rank 1
- A is a reduced p-group

Some notation

For $\alpha < \omega_1$ we let:

- $p^\alpha A$ be the subgroup of elements of p-height α
- $A^\alpha = p^{\omega\alpha} A$ be the α-th Ulm subgroup of A
- $L_\alpha(A)$ be the Polish group $\varprojlim_{\beta<\omega\alpha} A/p^\beta A$
- $E_\alpha(A)$ be the group with a Polish cover such that

$$0 \longrightarrow A/p^{\omega\alpha} A \longrightarrow L_\alpha(A) \longrightarrow E_\alpha(A) \longrightarrow 0$$

  is a Borel-definable exact sequence

Say that A is bounded if, for some $n < \omega$,

$$p^n A = \{0\}$$

or, equivalently,

$$L_1(A) = A.$$

A particular case of the main theorem

Theorem

Define α to be the least countable ordinal such that A^α is bounded.

- *If $\alpha = 0$, then $\{0\}$ is closed in $\mathrm{Ext}(C, A)$.*
- *If $\alpha \geq 1$, then the Borel rank of $\{0\}$ in $\mathrm{Ext}(C, A)$ is $\alpha + 2$.*

Solecki subgroups and Ulm subgroups

Lemma

For $\alpha < \omega_1$, $\mathrm{Ext}(C, A)^{1+\alpha} = s_\alpha(\mathrm{Ext}(C, A))$

In particular, we have that

$$\mathrm{Ext}(C, A)^1 = s_0(\mathrm{Ext}(C, A))$$

is the closure of $\{0\}$ in $\mathrm{Ext}(C, A)$.

The bounded case

Lemma (Mac Lane, 1960)

We have a Borel-definable exact sequence

$$0 \longrightarrow \mathrm{Ext}\left(C, A^{1}\right) \longrightarrow \mathrm{Ext}(C, A)^{1} \longrightarrow \mathrm{Hom}(C, E_{1}(A)) \longrightarrow 0$$

If A is bounded, then $L_1(A) = A$ and $E_1(A) = 0$ and $A^1 = 0$ and hence

$$0 \longrightarrow 0 = \mathrm{Ext}\left(C, A^{1}\right) \longrightarrow \mathrm{Ext}(C, A)^{1} \longrightarrow \mathrm{Hom}(C, E_{1}(A)) = 0 \longrightarrow 0$$

This shows that $s_0(\mathrm{Ext}(C, A)) = \mathrm{Ext}(C, A)^1 = 0$.

Hence $\{0\}$ is closed in $\mathrm{Ext}(C, A)$.

The unbounded case

Suppose that A is unbounded.

Define β to be the least ordinal such that $A^{1+\beta} = p^{\omega(1+\beta)}A$ is bounded

We need to show that $1 + \beta + 2$ is the Borel rank of $\{0\}$ in $\mathrm{Ext}(C, A)$.

Corollary

If $\rho(G)$ is a successor, then the Borel rank of $\{0\}$ in G is $1 + \rho(G) + 1$.

It suffices to prove that $\beta + 1$ is the Solecki rank of $\mathrm{Ext}(C, A)$.

This means that $s_{\beta+1}(\mathrm{Ext}(C, A)) = \{0\}$ and $s_{\beta}(\mathrm{Ext}(C, A)) \neq \{0\}$.

A Borel-definable exact sequence

Lemma (Nunke, 1967)

For every countable ordinal α, we have a Borel-definable exact sequence

$$0 \longrightarrow \mathrm{Ext}\left(C, A^{1+\alpha}\right) \longrightarrow \mathrm{Ext}(C, A)^{1+\alpha} \longrightarrow \mathrm{Hom}(C, E_{1+\alpha}(A)) \longrightarrow 0$$

The $(\beta+1)$-st Solecki subgroup

Since $A^{1+\beta}$ is bounded, we have that $p^n(p^{\omega(1+\beta)}A)$ for some $n < \omega$. Thus

$$L_{1+\beta+1}(A) = \varprojlim_{n<\omega} \frac{A}{p^n(p^{\omega(1+\beta)}A)} = A$$

and hence

$$A^{1+\beta+1} = 0 \quad \textit{and} \quad E_{1+\beta+1}(A) = 0$$

Thus

$$0 = \mathrm{Ext}(C, A^{1+\beta+1}) \longrightarrow \mathrm{Ext}(C, A)^{1+\beta+1} \longrightarrow \mathrm{Hom}(C, E_{1+\beta+1}(A)) = 0$$

and hence

$$s_{\beta+1}(\mathrm{Ext}(C, A)) = \mathrm{Ext}(C, A)^{1+\beta+1} = 0$$

The β-th Solecki subgroup

At the same time we have $p^{\gamma} A$ is nonzero for $\gamma < \omega(1+\beta)$ and hence

$$E_{1+\beta}(A) \text{ is nonzero and divisible}$$

and

$$\mathrm{Hom}\left(C, E_{1+\beta}(A)\right) \neq 0$$

From the Borel-definable exact sequence

$$0 \longrightarrow \mathrm{Ext}(C, A^{1+\beta}) \longrightarrow \mathrm{Ext}(C, A)^{1+\beta} \longrightarrow \mathrm{Hom}(C, E_{1+\beta}(A)) \longrightarrow 0$$

we obtain

$$s_{\beta}\left(\mathrm{Ext}(C, A)\right) = \mathrm{Ext}\left(C, A\right)^{1+\beta} \neq 0$$

This concludes the proof that $\rho(\mathrm{Ext}(C, A)) = \beta + 1$.