Algorithmic randomness and Bayesian convergence

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Monday, December 6, 2021
Midwest Computability Seminar

Overview

Much recent work in algorithmic randomness has concerned characterizations of randomness notions in terms of effectivizations of almost-everywhere convergence theorems in analysis and probability theory.

In our project, we study from this perspective results that are part of the basic toolkit of Bayesian epistemologists.

I will focus on certain martingale convergence theorems that form one of the cornerstones of Bayesian epistemology and that fall under the general umbrella of "Bayesian convergence-to-the-truth results".

General lesson: for Bayesian agents with computable priors, a certain type of inductive success is attainable exactly on the algorithmically random data streams.

The washing out of priors

In philosophy, Bayesianism is a family of views under which probability and degrees of belief become aligned.

Depending on the purpose at hand, this might make prior probability assignments look insufficiently objective.

Convergence-to-the-truth results are supposed to address this:

> *For the Bayesian, concerned as he is to deal with the real world of ordinary and scientific experience, the existence of a systematic method for reaching agreement is important. [...] The well-designed experiment is one that will swamp divergent prior distributions with the clarity and sharpness of its results, and thereby render insignificant the diversity of prior opinion.*
>
> [Suppes (1966), p. 204.]

The basic set-up

We work with probability spaces of the form $(2^{\mathbb{N}}, \mathcal{B}(2^{\mathbb{N}}), \mu)$.

Let $f : 2^{\mathbb{N}} \to \mathbb{R}$ be a random variable, and let $\mathbb{E}_\mu[f]$ denote the expectation of f with respect to μ. Then, f is integrable (or in L^1) if $\mathbb{E}_\mu[|f|] < \infty$.

For each n, let $\mathfrak{F}_n$ denote the sub-σ-algebra of $\mathcal{B}(2^{\mathbb{N}})$ generated by the clopens $[\sigma]$ associated to strings $\sigma \in 2^{<\mathbb{N}}$ of length n.

Intuitively, the filtration $\{\mathfrak{F}_n\}_{n\in\mathbb{N}}$ represents the possible *bodies of evidence* available at each stage n of the learning process.

Conditional expectation

$\mathbb{E}_\mu[f \mid \mathfrak{F}_n](X)$ is the expected value of f given knowledge in $\mathfrak{F}_n$.

Informally, $\mathbb{E}_\mu[f \mid \mathfrak{F}_n](X)$ is the *best estimate* of f's value on input $X \in 2^{\mathbb{N}}$ after having observed the first n digits of X.

We work with the following version of the conditional expectation:

$$\mathbb{E}_\mu[f \mid \mathfrak{F}_n](X) = \begin{cases} \frac{1}{\mu([X \upharpoonright n])} \int_{[X \upharpoonright n]} f \, d\mu & \text{if } \mu([X \upharpoonright n]) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$\{\mathbb{E}_\mu[f \mid \mathfrak{F}_n]\}_{n \in \mathbb{N}}$ is a martingale:

$$\mathbb{E}_\mu[\mathbb{E}_\mu[f \mid \mathfrak{F}_{n+1}] \mid \mathfrak{F}_n] = \mathbb{E}_\mu[f \mid \mathfrak{F}_n]$$

by the tower property of conditional expectations.

Conditional expectation and martingales

Again, we work with the following version of the conditional expectation:

$$\mathbb{E}_\mu[f \mid \mathfrak{F}_n](X) = \begin{cases} \frac{1}{\mu([X \restriction n])} \int_{[X \restriction n]} f \, d\mu & \text{if } \mu([X \restriction n]) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define $M : 2^{<\mathbb{N}} \to \mathbb{R}$ by defining, for σ of length n:

$$M(\sigma) = \mathbb{E}_\mu[f \mid \mathfrak{F}_n](\sigma^\frown \overline{0}) = \begin{cases} \frac{1}{\mu([\sigma])} \int_{[\sigma]} f \, d\mu & \text{if } \mu([\sigma]) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, this satisfies the following, which, in computability theory, we usually take to be the defining property of a martingale:

$$M(\sigma)\mu([\sigma]) = M(\sigma 0)\mu([\sigma 0]) + M(\sigma 1)\mu([\sigma 1])$$

Conditional probability

Again, we work with the following version of the conditional expectation:

$$\mathbb{E}_\mu[f \mid \mathfrak{F}_n](X) = \begin{cases} \frac{1}{\mu([X \restriction n])} \int_{[X \restriction n]} f \, d\mu & \text{if } \mu([X \restriction n]) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Conditional probabilities are of course a special case.

When f is the indicator function I_A of a measurable event A,

$$\mathbb{E}_\mu[I_A \mid \mathfrak{F}_n](X) = \begin{cases} \frac{\int_{[X \restriction n]} I_A d\mu}{\mu([X \restriction n])} = \frac{\mu(A \cap [X \restriction n])}{\mu([X \restriction n])} & \text{if } \mu([X \restriction n]) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

And $\frac{\mu(A \cap [X \restriction n])}{\mu([X \restriction n])} = \mu(A \mid [X \restriction n])$ is simply the probability of event A, conditional on the first n digits of X.

Lévy's Upward Martingale Convergence Theorem

Here is a classical theorem of Lévy on martingale convergence (stated in the context of probability spaces of the form $(2^{\mathbb{N}}, \mathcal{B}(2^{\mathbb{N}}), \mu)$):

Theorem (Lévy's Upward Theorem, Lévy [1937])

Let $f : 2^{\mathbb{N}} \to \mathbb{R}$ be integrable, relative to $(2^{\mathbb{N}}, \mathcal{B}(2^{\mathbb{N}}), \mu)$. Then,

$$\lim_{n \to \infty} \mathbb{E}_\mu[f \mid \mathfrak{F}_n] = f$$

μ-almost everywhere, and

$$\mathbb{E}_\mu[f \mid \mathfrak{F}_n] \xrightarrow{L^1(\mu)} f.$$

Epistemic interpretation of Lévy's Upward Theorem

Sequences in $2^{\mathbb{N}}$ are data streams.

The random variable f represents a quantity that a Bayesian agent with prior μ is trying to estimate—a quantity whose value depends on the observed data stream.

When f is the indicator function of a measurable event, then the quantity to be estimated is the truth value of that event.

The conditional expectation $\mathbb{E}_{\mu}[f \mid \mathfrak{F}_n](X)$ represents the Bayesian agent's beliefs about the value of f at the n-th stage of the learning process, after n observations have been made.

From this perspective, Lévy's Upward Theorem establishes that, with probability one (relative to the agent's prior), the agent's beliefs converge to the truth (i.e., the correct value of f) with increasing evidence.

Computable probability measures

In effectivizing Lévy's Upward Theorem, we work with arbitrary computable probability measures μ.

For some purposes, distinct computable probability measures do not differ much.

For instance, Kautz showed that Turing degrees of MLR's are not different among computable atomless measures [Kautz, 1991, Corollary IV.3.18, p. 69]. But our questions here are about elements of Cantor space, rather than their Turing degree.

And, of course, an extension of the Borel Isomorphism Theorem says that any two uncountable atomless Borel probability spaces are Borel isomorphic [Kechris, 1995, p. 116]. But the Borel Isomorphism Theorem is non-effective (e.g., not all uncountable Π^0_1-classes have computable members).

Computable probability measures

Most of what we do works for all computable probability measures.

This is important for our intended philosophical applications, since it is natural to interpret computable probability measures as the priors (the initial degrees of belief) of computationally limited Bayesian agents.

Sometimes, in what follows, we restrict to atomless computable probability measures.

We will end with an example where the only proof we have is for all computable Bernoulli measures.

General shape of our results

Theorem

Let $X \in 2^{\mathbb{N}}$ and μ a computable probability measure. The following are equivalent:

1. *X is ⎵⎵⎵⎵ μ-random;*
2. *for all ⎵⎵⎵⎵ f,*

$$\lim_{n\to\infty} \mathbb{E}_\mu[f \mid \mathfrak{F}_n](X) = f(X).$$

Epistemic interpretation

For computable Bayesian agents (i.e., agents with computable priors), beliefs, in the form of the agent's best estimates of the true value of a random variable, align with the truth in the limit, under appropriate algorithmic randomness assumptions.

Different effectivity constraints on random variables track how difficult the values of these random variables are to approximate.

For natural classes of effective random variables, we can pinpoint the collection of data streams that guarantee convergence to the truth for all random variables in that class.

In each case, the collection of truth-conducive data streams coincides with a specific algorithmic randomness notion.

L^1-computable functions

Definition (L^1-computable function)

Let μ be a computable probability measure.

1. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of measurable functions converges *fast* in $L^1(\mu)$ to a measurable function f if $||f_n - f||_{L^1(\mu)} \leq 2^{-n}$ for all $n \in \mathbb{N}$. We write that $f_n \to f$ fast in $L^1(\mu)$.
2. A function $f :\subseteq 2^{\mathbb{N}} \to \mathbb{R}$ is $L^1(\mu)$-*computable* if there is a computable sequence $\{f_n\}_{n\in\mathbb{N}}$ of rational-valued step functions such that $f_n \to f$ fast in $L^1(\mu)$. Such a sequence is said to be a *witness* to the $L^1(\mu)$-computability of f.

Schnorr randomness and Lévy's Upward Theorem

Theorem

Let $X \in 2^{\mathbb{N}}$ and μ a computable probability measure. The following are equivalent:

1. *X is Schnorr μ-random;*
2. *for all $L^1(\mu)$-computable functions $f :\subseteq 2^{\mathbb{N}} \to \mathbb{R}$ with witness $\{f_m\}_{m \in \mathbb{N}}$,*
$$\lim_{m \to \infty} f_m(X) \text{ exists and is finite,}$$
and
$$\lim_{n \to \infty} \mathbb{E}_\mu[f \mid \mathfrak{F}_n](X) = \lim_{m \to \infty} f_m(X).$$

Integral tests for Schnorr randomness

Let μ be a computable probability measure. An *integral test for Schnorr μ-randomness* is a non-negative lower semi-computable function $f : 2^{\mathbb{N}} \to \overline{\mathbb{R}}$ with $\int_{2^{\mathbb{N}}} f \, d\mu$ computable.

Theorem (Miyabe [2013])

Let $X \in 2^{\mathbb{N}}$ and μ a computable probability measure. The following are equivalent:

1. *X is Schnorr μ-random;*
2. *$f(X) < \infty$ for all integral tests for Schnorr μ-randomness.*

Schnorr randomness and Lévy's Upward Theorem

Theorem

Let $X \in 2^{\mathbb{N}}$ and μ a computable probability measure. The following are equivalent:

1. *X is Schnorr μ-random;*
2. *for all integral tests for Schnorr μ-randomness $f : 2^{\mathbb{N}} \to \overline{\mathbb{R}}$,*

$$\lim_{n \to \infty} \mathbb{E}_{\mu}[f \mid \mathfrak{F}_n](X) = f(X) < \infty.$$

Relation to Pathak, Rojas, and Simpson's work

Our result is the Cantor-space analogue of Pathak et al.'s [2014] characterization of Schnorr randomness in terms of the Lebesgue Differentiation Theorem in Euclidean space for the uniform measure:

Theorem (Pathak et al. [2014])

Let $X \in [0,1]^n$. The following are equivalent, where λ is the Lebesgue measure on $[0,1]^n$:

1. *X is Schnorr λ-random;*
2. *for all $L^1(\lambda)$-computable $f :\subseteq [0,1]^n \to \mathbb{R}$ with witness $\{f_m\}_{m\in\mathbb{N}}$,*

$$\lim_{m\to\infty} f_m(X) \text{ exists and is finite,}$$

and

$$\lim_{Q\to X} \frac{\int_Q f(X')\,dX'}{\lambda(Q)} = \lim_{m\to\infty} f_m(X).$$

Relation to Rute's work

Theorem (Rute [2012])

Let $X \in 2^{\mathbb{N}}$ and μ a computable probability measure. The following are equivalent:

1. *X is Schnorr μ-random;*
2. *for all uniformly $L^1(\mu)$-computable martingales M_k which converge in $L^1(\mu)$ to an L^1-computable function, $\lim_{k \to \infty} M_k(X)$ converges.*

In motivating his work, Rute pointed out that "algorithmic randomness is more concerned with success than convergence."

We focus not merely on convergence *per se*, but on convergence to the correct value, since we need this information for our intended philosophical applications.

Relation to Rute's work

Theorem (Rute [2012])

Let $X \in 2^{\mathbb{N}}$ and μ a computable probability measure. The following are equivalent:

1. *X is Schnorr μ-random;*
2. *for all non-negative $L^1(\mu)$-computable martingales M_k with respect to the filtration $\mathfrak{F}_k$ such that $M_k \to 0$ a.e., $\lim_{k\to\infty} M_k(X) = 0$.*

But if $M_n \to 0$ a.e., then there is no computable $L^1(\mu)$-function f such that $M_n(X) = \mathbb{E}_\mu[f \mid \mathfrak{F}_n](X)$. (This is because the hypothesis $M_n \to 0$ a.e. on the martingale forces its measure representation to be mutually singular with respect to the measure).

Density randomness

Definition (Density randomness)

Let μ be a computable measure. A sequence $X \in 2^{\mathbb{N}}$ is *density μ-random* if and only if it is Martin-Löf μ-random, as well as a dyadic density-one point relative to μ: i.e.,

$$\liminf_{n \to \infty} \frac{\int_{[X \upharpoonright n]} I_C d\mu}{\mu([X \upharpoonright n])} = 1$$

for all Π^0_1 classes C such that $X \in C$.

Weakly L^1-computable functions

Definition (Weakly L^1-computable function)

Let μ be a computable probability measure. Then, a function $f :\subseteq 2^{\mathbb{N}} \to \mathbb{R}$ is *weakly $L^1(\mu)$-computable* if there is a computable sequence $\{f_n\}_{n\in\mathbb{N}}$ of rational-valued step functions such that

- $f_n \xrightarrow{L^1(\mu)} f$ and
- $\sum_{n\in\mathbb{N}} ||f_{n+1} - f_n||_{L^1(\mu)} < \infty$ (i.e., the sequence $\{f_n\}_{n\in\mathbb{N}}$ has bounded $L^1(\mu)$-variation.

Density randomness and Lévy's Upward Theorem

Theorem

Let $X \in 2^{\mathbb{N}}$ and μ a computable, strictly positive and atomless probability measure. The following are equivalent:

1. *X is density μ-random;*
2. *for all weakly $L^1(\mu)$-computable functions $f :\subseteq 2^{\mathbb{N}} \to \mathbb{R}$ with witness $\{f_m\}_{m\in\mathbb{N}}$, we have that*

$$\lim_{n\to\infty} f_n(X) \text{ exists and is finite,}$$

and

$$\lim_{k\to\infty} \mathbb{E}_\mu[f \mid \mathfrak{F}_k](X) = \lim_{n\to\infty} f_n(X).$$

Integral tests for Martin-Löf randomness

Let μ be a computable probability measure. An *integral test for Martin-Löf μ-randomness* is a non-negative lower semi-computable function $f : 2^{\mathbb{N}} \to \overline{\mathbb{R}}$ with $\int_{2^{\mathbb{N}}} f \, d\mu < \infty$.

Theorem (Levin [1976])

Let $X \in 2^{\mathbb{N}}$ and μ a computable probability measure. The following are equivalent:

1. *X is Martin-Löf μ-random;*
2. *$f(X) < \infty$ for all integral tests for Martin-Löf μ-randomness.*

Density randomness and Lévy's Upward Theorem

Theorem

Let $X \in 2^{\mathbb{N}}$ and μ a computable, strictly positive and atomless probability measure. The following are equivalent:

1. *X is density μ-random;*
2. *for all integral tests for Martin-Löf μ-randomness $f : 2^{\mathbb{N}} \to \overline{\mathbb{R}}$,*

$$\lim_{n\to\infty} \mathbb{E}_{\mu}[f \mid \mathfrak{F}_n](X) = f(X) < \infty.$$

Relation to Miyabe, Nies, and Zhang's work

Theorem (Miyabe et al. [2016])

Let $X \in [0,1]$. The following are equivalent, where λ is the Lebesgue measure on $[0,1]$:

1. *X is density λ-random;*
2. *for all integral tests for Martin-Löf λ-randomness $f : [0,1] \to \overline{\mathbb{R}}$,*

$$\lim_{Q \to X} \frac{\int_Q f(X')\, dX'}{\lambda(Q)} = f(X).$$

Gaifman-Snir randomness

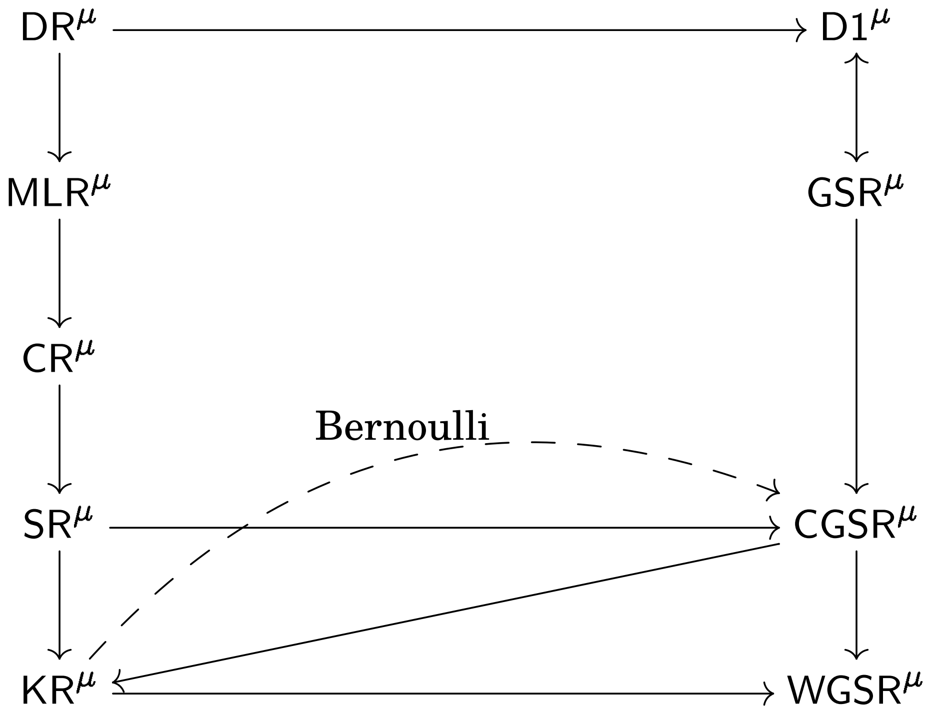
The first authors to suggest a bridge between randomness and Bayesian epistemology were Gaifman and Snir [1982], who focused on convergence to the truth in the context of indicator functions of measurable events.

Definition (Gaifman-Snir randomness)

Let $X \in 2^{\mathbb{N}}$ and μ a computable probability measure. Then,

- X is *Gaifman-Snir μ-random* if $\mu([X \upharpoonright n]) > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathbb{E}_\mu[I_U \mid \mathfrak{F}_n](X) = I_U(X)$ for all Σ^0_1 classes U;
- X is *computably Gaifman-Snir μ-random* if $\mu([X \upharpoonright n]) > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathbb{E}_\mu[I_U \mid \mathfrak{F}_n](X) = I_U(X)$ for all Σ^0_1 classes U with $\mu(U)$ computable;
- X is *weakly Gaifman-Snir μ-random* if $\mu([X \upharpoonright n]) > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathbb{E}_\mu[I_U \mid \mathfrak{F}_n](X) = I_U(X)$ for all Δ^0_1 classes U.

Gaifman-Snir randomness



$\mathsf{KR}^\mu \subseteq \mathsf{CGSR}^\mu$ for computable Bernoulli measures

Proposition

Let μ be a computable Bernoulli measure. Then, $\mathsf{KR}^\mu \subseteq \mathsf{CGSR}^\mu$.

Proof.

Suppose that $X \in \mathsf{KR}^\mu$ and that U is Σ^0_1 with $\mu(U)$ computable. We want to show that $\lim_{n\to\infty} \mathbb{E}_\mu[I_U \mid \mathfrak{F}_n](X) = I_U(X)$.

This is trivial if $X \in U$, so suppose that $X \notin U$. We want to show that $\lim_{n\to\infty} \mathbb{E}_\mu[I_U \mid \mathfrak{F}_n](X) = 0$.

Suppose not. Then there is a rational $\epsilon > 0$ such that there are infinitely many n with $\mathbb{E}_\mu[I_U \mid \mathfrak{F}_n](X) > \epsilon$.

Since $\mu(U)$ is computable, there is a computable function $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \geq n$ and $\mathbb{E}_\mu[I_U \mid \mathfrak{F}_{f(n)}](X) > \epsilon$.

$\mathsf{KR}^\mu \subseteq \mathsf{CGSR}^\mu$ for computable Bernoulli measures

Let C be the Π^0_1 class of the $Y \in 2^{\mathbb{N}}$ such that $Y \notin U$ and, for all $n \in \mathbb{N}$, $\mathbb{E}_\mu[I_U \mid \mathfrak{F}_{f(n)}](Y) > \epsilon$.

If $\mu(C) = 0$, then we would have that X, being in KR^μ, is not in it. Hence, $\mu(C) > 0$.

Now we verify that the usual proof that Π^0_1-classes of positive measure contain tails of all MLR's extends to all Bernoulli measures.

Since $\mu(C) > 0$, its complement $\overline{C}$ is a Σ^0_1 class with $\mu(\overline{C}) < 1$.

Let $S = \{\sigma_0, \sigma_1, ...\} \subseteq 2^{<\mathbb{N}}$ be a prefix-free set such that $\overline{C} = [S]$.

Define the following sequence of sets of strings:

- $S_0 := S$ and,
- given S_n, $S_{n+1} := \{\sigma\tau \in 2^{<\mathbb{N}} : \sigma \in S_n \text{ and } \tau \in S\}$.

$\mathsf{KR}^\mu \subseteq \mathsf{CGSR}^\mu$ for computable Bernoulli measures

For each n, let $V_n = [S_n]$. Then, V_n is a sequence of uniformly Σ^0_1 classes.

Moreover, $\mu(V_0) = \mu(\overline{C}) < 1$ and, for all $n \geq 1$,

$$\begin{aligned}
\mu(V_n) &= \mu([S_n]) \\
&= \sum_{\sigma \in S_{n-1}, \tau \in S} \mu([\sigma\tau]) \\
&= \sum_{\sigma \in S_{n-1}, \tau \in S} \mu([\sigma])\mu([\sigma\tau] \mid [\sigma]) \\
& \sum_{\sigma \in S_{n-1}, \tau \in S} \mu([\sigma])\mu([\tau]) \\
&= \mu([S_{n-1}])\mu([S]) \\
&= \mu([S])^{n+1}
\end{aligned}$$

$\mathsf{KR}^\mu \subseteq \mathsf{CGSR}^\mu$ for computable Bernoulli measures

Let $q < 1$ a rational with $\mu([S]) < q$. For each n, let k_n be the least m such that $q^{m+1} \leq 2^{-n}$ and let $V'_n = V_{k_n}$.

Since k_n can be found computably in n, V'_n is a sequence of uniformly Σ^0_1 classes. Moreover, $\mu(V'_n) \leq 2^{-n}$. Hence, V'_n is a Martin-Löf μ-test.

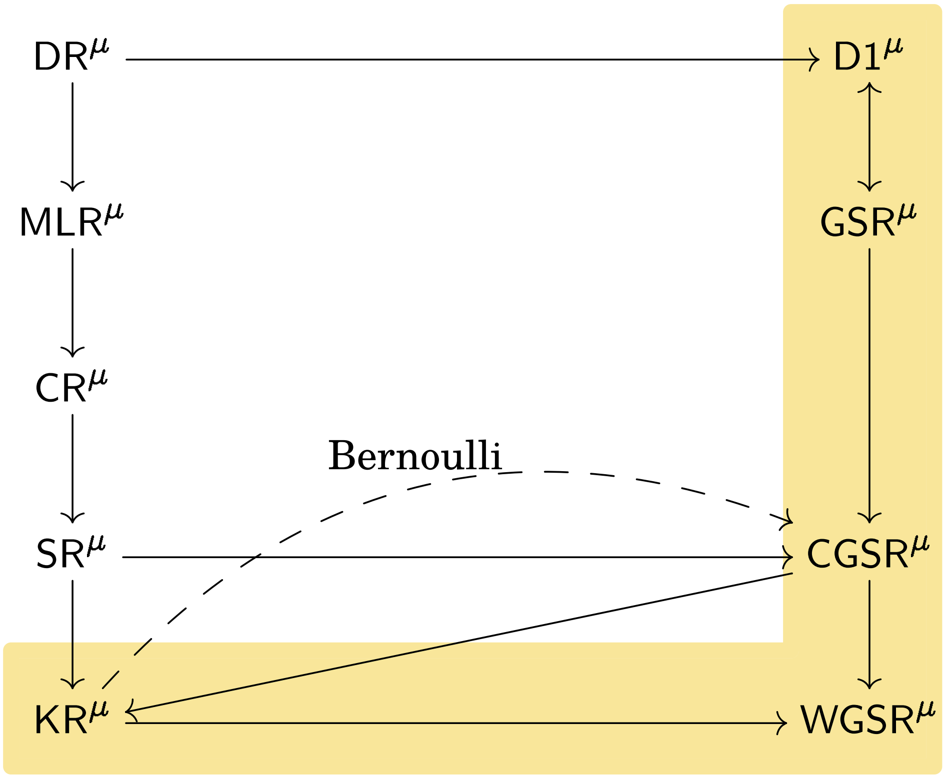
Let $Y \in \mathsf{MLR}^\mu$ and suppose that, for all $\sigma \in 2^{<\mathbb{N}}$ and $Z \in 2^{\mathbb{N}}$, if $Y = \sigma Z$, then $Z \in \overline{C} = [S]$. Then, for all n, $Y \in [S_n] = V_n$.

Hence, $Y \in \bigcap_{n \in \mathbb{N}} V'_n$. But this contradicts the assumption that $Y \in \mathsf{MLR}^\mu$. So, there must be some $\sigma \in 2^{<\mathbb{N}}$ and $Z \in 2^{\mathbb{N}}$ with $Y = \sigma Z$ and $Z \in C$.

Since $Y \in \mathsf{MLR}^\mu$, so is Z. Hence, $Z \in \mathsf{SR}^\mu$.

By the definition of C, $Z \notin U$, but it is not the case that $\lim_{n\to\infty} \mathbb{E}_\mu[I_U \mid \mathfrak{F}_n](Z) = 0$. And this contradicts the fact that $\mathsf{SR}^\mu \subseteq \mathsf{CGSR}^\mu$.

Gaifman-Snir randomness, genericity, and Bayesian immodesty



Bayesian epistemology from a computability-theoretic perspective

Bayesian convergence-to-the-truth results are not the only results relevant to Bayesian epistemology that can be studied from the perspective of algorithmic randomness.

As part of our project, we are now focusing on Bayesian merging-of-opinions results, which are standardly taken to establish that distinct Bayesian agents with sufficiently compatible priors are guaranteed to reach inter-subjective agreement with probability one.

The overall goal is two-fold:

- develop a computability-theoretic approach to Bayesian epistemology;
- study the notions of randomness/genericity that emerge from natural effectivizations of convergence results relevant to Bayesian epistemology.

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