# Parametrizing the Ramsey Theory of Vector Spaces – Iian Smythe (Michigan)

Ramsey theory (idea): If you partition a "large enough" structure into a "small enough" number of pieces, one of those pieces should be "large".

Fix a ctbly infinite dim. vector space $E$ over a ctbl field $F$, with basis $(e_n)$.

E.g. $E = \bigoplus_n F$.

Terminology: "vector" means nonzero vector
"subspace" means $\infty$-dim (linear) subspace

For a vector $v = \sum a_n e_n$, its support is

$$\mathrm{supp}(v) = \{n \in \omega : a_n \neq 0\}.$$

Write: $v < w$ if $\max(\mathrm{supp}(v)) < \min(\mathrm{supp}(w))$.

Say that $(v_n)$ is a block sequence if $v_n < v_{n+1}\ \forall n$.

Fact: Every subspace contains a block sequence.

$E^{[\infty]}$ = space of all (infinite) block seq. in $E$

a Polish subspace of $E^{\omega}$.

For $X, Y \in E^{[\infty]}$, $X \preceq Y$ if $\langle X \rangle \subseteq \langle Y \rangle$.

span

( $X \preceq^* Y$ if $X/n \preceq Y$ for some $n$ )

$X$ "above $n$"

(if $|F| > 2$)

Fact: There is a partition of all vectors in $E$ into 2 pieces neither of which contains a subspace.

Fix $X \in E^{[\infty]}$.

Asymptotic Game $F[X]$

| | | | | | | |
|---|---|---|---|---|---|---|
| I | $n_0$ | | $n_1$ | | $n_2$ | ... |
| II | | $v_0 \in \langle X/n_0 \rangle$ | | $v_1 \in \langle X/n_1 \rangle$ | | $v_2$ |

Require $(v_n)$ is a block seq.

## Gowers Game $G[X]$

| | | | |
|---|---|---|---|
| I | $Y_0 \preceq X$ | $Y_1 \preceq X$ | $Y_2 \preceq X$ ... |
| II | $v_0 \in \langle Y_0 \rangle$ | $v_1 \in \langle Y_1 \rangle$ | |

Outcome: $(v_n) \in E^{[\infty]}$

Obs: If I has a strat. to play into some $\mathbb{A} \subseteq E^{[\infty]}$ in $F[X]$, they have one in $G[X]$.
Vice-versa for II.

Thm (Rosendal, 2010): If $\mathbb{A} \subseteq E^{[\infty]}$ is analytic, then $\exists X \in E^{[\infty]}$ s.t. either:
(1) I has a strat. in $F[X]$ to play out of $\mathbb{A}$.
OR
(2) II ........... $G[X]$ .... in to $\mathbb{A}$.

### A parametrized form:

Instead of $\mathbb{A} \subseteq E^{[\infty]}$, have $\{\mathbb{A}_t : t \in \mathbb{R}\}$.

Thm (S.) If $\mathbb{A} \subseteq \mathbb{R} \times E^{[\infty]}$ (analytic), then $\exists X \in E^{[\infty]}$ and a perfect $P \subseteq \mathbb{R}$ s.t. either:

can get the same strat. for all $t \in P$

(1) $\forall t \in P\,(\mathrm{I}$ has a strat. in $F[X]$ to play out of $\mathbb{A}_t)$

OR

(2) $\forall t \in P\,(\mathrm{II}$ has a strat. in $G[X]$ to play into $\mathbb{A}_t)$

where $\mathbb{A}_t = \{X : (t, X) \in \mathbb{A}\}$.

A consequence:

<u>Thm (S.)</u> For any analytic family $\mathcal{a} = \{T_t : t \in \mathbb{R}\}$ of linear trans. $E \to E$, there is a subspace $V \subseteq E$ and a perfect $P \subseteq \mathbb{R}$ s.t.

(A) $\forall t \in P\,(V \subseteq \ker(T_t))$,

OR

(B) $\forall t \in P\,(T_t \upharpoonright V$ is injective$)$.

<u>pf:</u> Let $\mathbb{A} = \{(t, (v_n)) \in \mathbb{R} \times E^{[\infty]} : T_t(v_0) \neq 0\}$. $\mathbb{A}$ is analytic. By prev. thm, get $X \in E^{[\infty]}$ and a perfect $P \subseteq \mathbb{R}$ s.t. either (1) or (2) holds.

Say (1) holds. So there is a strat. $\sigma$ for I in $F[X]$ to play out of $\mathbb{A}_t$, for every $t \in P$.

Let $n_0 = \sigma(\emptyset)$. Take $v \in \langle X/n_0 \rangle$, then $v$ is a valid response in $F[X]$ against $\sigma$.

But then $T_t(v) = 0$. Take $V = \langle X/n_0 \rangle$, so $V \subseteq \ker(T_t)$ $\forall t \in P$, proving (A).

Say (2) holds. Let $t \in P$.

<u>Claim</u>: $T_t \restriction \langle X \rangle$ has finite-dim. kernel.

Why? If not, $\exists Y \in E^{[\infty]}$ s.t. $\langle Y \rangle \subseteq \ker(T_t \restriction \langle X \rangle)$.

So, $Y \preceq X$. Then I can play $Y$ on their 1st move in $G[X]$, and for any response $v \in \langle Y \rangle$ by II, $T_t(v) = 0$. This contradicts II having a strat. into $\mathbb{A}_t$.

By the claim, $P = \bigcup_n P_n$

$$P_n = \{t \in P : \ker(T_t \restriction \langle X \rangle) \subseteq \langle e_0, \dots, e_n \rangle\}.$$

By PSP, there is some $n_0$ and some perfect $Q \subseteq P_{n_0}$.

Now: $V = \langle X/n_0 \rangle$ and $Q$ witness (B). $\square$