Zero–one laws for finitely presented structures

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Zero-one law for graphs

Erdos-Renyi random graphs $G(n,p)$:
vertices = $\{0, 1, \cdots, n-1\}$
each pair of vertices has an edge with probability p

Theorem (Fagin, '76)

Let φ be a first-order sentence in the language of graphs. Then $Pr(G(n,p) \models \varphi) \to 0$ or 1 as $n \to \infty$. Furthermore, the $Pr(G(n,p) \models \varphi) \to 1$ iff the random graph $G(\omega, p) \models \varphi$.

Knight's conjecture

- Gromov '87: Definition of *random groups*
- Random groups are infinite, torsion-free, non-abelian, hyperbolic, one-ended, and has lots of free subgroups.
- Tarski's problem (Sela '06; Kharlampovich and Myasnikov '06): non-abelian free groups of different ranks are elementarily equivalent

Conjecture (Knight, '13)

A first-order sentence is true in a free group iff it is true in a random group.

Theorem (Kharlampovich and Sklinos, '21)

A universal first-order sentence is true in a free group iff it is true in a random group.

A toy example

$L = \{S(x), S^{-1}(x)\}$
T = "S and S^{-1} are inverse functions."

Consider a single generator a.
Random identities: $S^{\epsilon_1} \cdots S^{\epsilon_i}(a) = S^{\epsilon'_1} \cdots S^{\epsilon'_j}(a) \Leftrightarrow S^k(a) = a$
$\langle \overline{a} \mid S^k(a) = a \rangle$: the "freest" structure where $S^k(a) = a$
What happens when $i, j \to \infty$?

Lemma

Over T, every sentence is equivalent to a Boolean combination of:

- *"there are m disjoint cycles of size n"*
- *"there is a chain of length $\geq n$"*

Theorem

For every sentence φ, φ is true in a 1-generated random structure iff it is true in the 1-generated free structure.

Algebraic varieties and presentations

We consider algebraic varieties/equational classes in the sense of universal algebra.

Definition (Birkhoff, '35)

- A language is *algebraic* if it contains no relation symbols.
- *algebraic variety*: a class of L-structures axiomatized by some sentences of the form $\forall \overline{x}\ t(\overline{x}) = s(\overline{x})$.
- *free structure* $\langle \overline{a} \mid \varnothing \rangle$: the term algebra modulo the axioms
- *presentation* $\langle \overline{a} \mid r \rangle$: the "freest" structure where $r \equiv u(\overline{a}) = v(\overline{a})$

Example

Groups and rings are algebraic varieties.

Random presentation

Definition

- $P_\ell(\varphi)$ = the probability that $\langle \overline{a} \mid r\rangle \vDash \varphi$ for a randomly chosen r.
- *a random structure in V satisfies* φ if its *limiting density* $\lim_{\ell\to\infty} P_\ell(\varphi) = 1$.
- *V satisfies the zero–one law* if for every sentence φ, $\lim_{\ell\to\infty} P_\ell(\varphi) \in \{0, 1\}$.
- *V satisfies the strong zero–one law* if for every sentence φ, $\lim_{\ell\to\infty} P_\ell(\varphi) = 1$ iff $\langle \overline{a} \mid \varnothing\rangle \vDash \varphi$.

Example

- This coincides with Gromov's random groups model.
- The variety with a pair of inverse functions satisfies the strong zero–one law.

General question

Question

Classify the three possibilities:

- *the variety does not admit a limiting theory*
- *(weak) zero–one law: the variety admits a limiting theory but differs with the free structure*
- *strong zero–one law: the variety admits a limiting theory that agrees with the free structure*

In the variety with a pair of inverse functions:

1. Random identities cannot be detected locally
2. Every sentence is equivalent to a Boolean combination of local sentences

Some examples

Example

The variety with $L = \{f(x)\}$ and $T = \varnothing$ satisfies the 0–1 law, but the limiting theory differs from the theory of the free structure.

Some examples

Example

The variety with $L = \{f(x), g(x)\}$ and $T = \varnothing$ does not satisfy the 0–1 law.

Some examples

Example

In the variety with $L = \{S(x), S^{-1}(x)\}$ and $T = \{\forall x\ S(S^{-1}(x)) = S^{-1}(S(x)) = x\}$, but with *two* identities, the sentence $\varphi = \forall x\ S(x) = x$ does not have limiting density.

In the variety with a pair of inverse functions:

1. Random identities cannot be detected locally
2. Every sentence is equivalent to a Boolean combination of local sentences

Gaifman's Locality Theorem

Definition

Let A be a relational structure. The *Gaifman graph* of A is the graph with $V = A$ and $(a, b) \in E$ if there is some R with $R(\overline{x})$ and $a, b \in \overline{x}$.

Let $B_r(\overline{x})$ be the r-neighborhood of $\overline{x}$. Then $y \in B_r(\overline{x})$ is definable in A. We write $\varphi^{(r)}(\overline{x})$ if all quantifiers are $\exists y \in B_r(\overline{x})$ or $\forall y \in B_r(\overline{x})$.

Theorem (Gaifman Locality Theorem, '82)

Let L be a relational language. Then every sentence is equivalent to a Boolean combination of sentences of the form

$$\exists v_1, \cdots, v_s \left(\bigwedge_i \alpha_i^{(r)}(v_i) \wedge \bigwedge_{i<j} d(v_i, v_j) > 2r \right).$$

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For a language with only unary functions, think of the structures as directed graphs.

$\alpha_i^{(r)}(v_i)$: formulas that describes the r-ball around v_i

$d(x, y)$: the distance function of the graph

Bijective varieties

We consider structures in the language $\{f_1, f_1^{-1}, \cdots, f_n, f_n^{-1}\}$.

Example

T = "f_i, f_j commute" and "f_i^{-1} is the inverse of f_i".
This variety satisfies the strong 0–1 law.
This corresponds to the variety of n-generated abelian groups, which does not satisfy the 0–1 law.

Example

T = "f_i^{-1} is the inverse of f_i".
This variety satisfies the strong 0–1 law.
This corresponds to the variety of n-generated groups.

Bijective varieties

Theorem

Let $L = \{f_1, \ldots, f_n\}$. If $T \vdash$ "f_i, f_j commute" and "f_i is bijective" and the free structure is infinite, then the variety with generators $a_1, \cdots, a_m$ satisfies the strong 0–1 law.
Furthermore, it satisfies the strong 0–1 law for sentences in the language $L' = L \cup \{a_1, \cdots, a_m\}$ where a_i are interpreted as the generators.

Question

- *What if we drop commutivity?*
- *Is bijectivity a necessary condition?*

Failure of the 0–1 law

Theorem

If there are two elements x_1 and x_2 in the free structure such that a random term equals x_i with a positive probability, then the variety does not satisfy the 0–1 law.

Example

Let $T = \{\forall x\ f^n(x) = x\}$.
A random structure in this variety is trivial with probability $\phi(n)/n$.

Naming generators

Theorem

Fix a variety in language L with generators $a_1, \ldots, a_m$. Let $L' = L \cup \{a_1, \ldots, a_m\}$ and T_F be the set of L'-sentences true in the free structure, and S be the set of L'-sentences true in a random structure. Then $T_F = S$ iff both of the following are satisfied:

- *S includes all sentences from T_F of the form $t(\bar{a}) \neq t'(\bar{a})$, and*
- *If for some $\varphi(x)$, $\varphi(t(\bar{a})) \in S$ for all closed terms $t(\bar{a})$, then $\forall x\, \varphi(x) \in S$.*

Question

Are there varieties which satisfies the strong 0–1 law for L but not for L'?

Further questions

Question

What happens if there are more generators or identities?
What if we allow constants?