Notes on Sacks Splitting Theorem

Rod Downey
Victoria University
Wellington, New Zealand
(Joint with Klaus Ambos-Spies, Martin Monath and Ng Keng Meng)

March, 2022

Motivation

- One of the fundamental results of computability theory is Sacks' Splitting Theorem:

Theorem (Sacks, 1963)

1. *If A is c.e. and noncomputable, then there exists a c.e. splitting $A_1 \sqcup A_2 = A$ with $A_1 |_T A_2$.*
2. *Indeed, if A is c.e. and $\varnothing <_T C \leq_T \varnothing'$ then there exists a c.e. splitting $A_1 \sqcup A_2 = A$ with $C \not\leq_T A_i$ for $i \in \{1, 2\}$.*

- This fundamental result
  1. Showed that there were no minimal c.e. degrees,
  2. Ushered in one form of the infinite injury method (although it is not an infinite injury argument, but finite injury of "unbounded type".)
  3. Was the basis of huge technical progress on the c.e degrees.

For Example

- The proof is the classic finite injury argument of this kind where A_i are built by meeting

$$R_{e,i} : \Phi_e(A_i) \neq A_j, (j \neq i).$$

- The idea is to preserve

$$\ell(e, i, s) \max\{x \mid \forall y < x \Phi_e(A_i) \neq A_j[s]\}.$$

- You preserve one side up to $\ell(e, s)$ and if this happens forever, then A will be computable.
- This method is at the heart, of, for example, the Sacks Density Theorem:

Theorem (Sacks 1963)

The computably enumerable degrees are dense.

- To my knowledge, every similar theorem for degree structures has worked in an analogous way.

For Example

Theorem (Robinson, 1971)

Everything c.e. If $C <_T A$ and C low, then $A = A_1 \sqcup A_2$ with $C \oplus A_1 |_T C \oplus A_2$. Hence every c.e. degree splits over each lesser low c.e. degree.

Robinson's Theorem was very influential in that it showed how to use "lowness+c.e." a theme we follow to this day.

Theorem (Lachlan, 1975)

There exist $\mathbf{c} < \mathbf{a}$ such that $\mathbf{a}$ does not split over $\mathbf{c}$.

That is, Sacks splitting and density cannot be combined. This legendary result of Lachlan affected the architecture of computability theory thereafter. E.g. definability, decidability etc. Invented the $\mathbf{0}'''$ method to prove this result. Harrington improved Lachlan's Theorem to have $\mathbf{a} = \mathbf{0}'$.

Re-examining this

- Lots of questions can be asked about the 60 year old result. One I "recently" looked at with Guohua Wu is the following.

Question

Is the natural analog for avoiding lower cones valid?

- The answer is no.

Theorem (Downey, Wu)

There are c.e. sets $B <_T A$ such that whenever $A_1 \sqcup A_2 = A$ is a c.e. splitting, then for some $i \in \{1, 2\}$, $A_i \leq_T B$.

- The proof is very complex.
- The degree analog is true because either $\mathbf{a}$ splits over $\mathbf{b}$, or $\mathbf{b}$ cups $\mathbf{a}_2$ to $\mathbf{a}$ for some $\mathbf{a}_2$ and we can then choose $\mathbf{b} < \mathbf{a}_1 < \mathbf{a}$ by Sacks's Density Theorem. (i.e. lower cone avoidance happens)

This Talk

- The question I examine here is

Question

How unbounded is the finite injury?

- What do we even mean by this?
- One possible way to do this is to use "Reverse Recursion Theory" (or maybe "Converse Computability Theory") by asking what amount of induction is needed for proving Sacks' Splitting Theorem in fragments of arithmetic.
- The setting is $P^- = PA -$ induction. Then $P^- + I\Sigma_n$ adds induction for Σ^0_n formulae, and $P^- + B\Sigma_n$ adds Σ^0_n bounding:

$$\forall x < a(\exists y \varphi(x, y) \to \exists b \forall x < a \exists y < b \varphi(x, y)),$$

where φ is Σ^0_n.

- Paris and Kriby (1978) showed

$$\mathbb{B}\Sigma_{n+1} \Rightarrow I\Sigma_n^0 \Rightarrow B\Sigma_n^0,$$

and all implications are proper.

Theorem (Mytilinaios 1989)

You can prove Sacks Splitting Theorem in $P^- + I\Sigma_1$.

Analog of earlier work in higher recursion theory $I\Sigma_1 \approx$ "Σ_1−admissible", and Shore's Thesis.

Theorem (Chong and Mourad, 1992)

1. *Friedberg-Muchnik can be proven in $P^- + B\Sigma_1$.*
2. *There is a model of $P^- + B\Sigma_1$ where Sacks Splitting Theorem fails. (both forms)*

- Here the interpretation is that the system with $B\Sigma_1$ corresponds to computably bounded injury. (See Chong, Li and Yang, BSL, 2014)
- There is a nice open question here of whether there is a theorem of classical computability theory equivalent to $B\Sigma^0_2$ over $I\Sigma^0_1$. Perhaps the hierarchy below might provide some new insights here. This is unexplored.

This Talk

- In this talk we will try to understand "finite injury of unbounded type" in classical terms.
- Perhaps we might see to understand how complex the splits A_1, A_2 must be. Certainly they can be chosen low, but can we quantify "how low" they can or must be? That is, in terms of computational power.
- To do this, we will use the Downey-Greenberg Hierarchy which is a classification tool for the complexity of computability constructions.
- This is based on "mind changes" for the functions sets A_i can compute. If X can compute a function with only complicated approximations given by the Limit Lemma, we regard it as computationally powerful. (Details below.)
- This is an old idea going back to Miller and Martin and the like, often outside of the Δ_2^0 degrees.
- In the Δ_2^0 degrees likely beginning with Martin's characterization of the high degrees in terms of domination.
- Revitalized in algorithmic randomness.

- In terms of this classification we have the following. (to be defined)

Theorem (Ambos-Spies, Downey, Monath, Ng)

If A is c.e. then A can always be split into a pair of totally ω^2-c.a. c.e. sets.

- Sacks' proof only gives ω^ω-c.a..
- Earlier Selwyn and I showed that this is tight:

Theorem (Downey and Ng, 2018)

There is a c.e. degree $\mathbf{a}$ such that if $\mathbf{a}_1 \vee \mathbf{a}_2 = \mathbf{a}$ then $\mathbf{a}_i$ is not totally ω-c.a. for $i \in \{1, 2\}$.

Truly Unbounded

- The upper cone avoiding version of Sacks Splitting Theorem is truly unbounded according to this classification tool because of the following:

Theorem (ADMN)

Let $\alpha < \epsilon_0$. Then there exists c.e. sets A and $\varnothing <_T C$ such that for all splittings $A_1 \sqcup A_2 = A$ of A, if A_1 is α-c.a. then $C \leq_T A_2$.

- Indeed, this holds for degrees:

Theorem (ADMN)

Let $\alpha < \epsilon_0$. Then there exists c.e. degrees $\mathbf{a}$ and $\mathbf{c}$ such that for all c.e degrees $\mathbf{a}_1, \mathbf{a}_2 \neq \mathbf{0}$ with $\mathbf{a}_1 \vee \mathbf{a}_2 = \mathbf{a}$, if $\mathbf{a}_1$ is α-c.a. then $\mathbf{c} \leq_T \mathbf{a}_2$.

The Hierarchy

- Recall

Theorem (Shoenfield 1959)

$A \leq_T \varnothing'$ iff there is a computable $f(\cdot,\cdot)$ with $A(x) = \lim_s f(x,s)$. This also holds for functions $g \leq_T \varnothing'$.

- Interestingly, also re-proven by Gold and by Putnam in 1965 (JSL).
- Recall that $g \leq_{wtt} \varnothing'$ if there is a procedure $\Phi^{\varnothing'} = g$ with use $\varphi(x)$ a computable function. The proof of Shoenfield's Limit Lemma shows that $g \leq_{wtt} \varnothing'$ iff it is ω-c.a.; that is there is a computable approximation $g(x,s)$ and computable h such that $\lim_s g(x,s) = g(x)$ and

$$|\{s \mid g(x,s+1) \neq g(x,s)\}| \leq h(x).$$

The Hierarchy

- The DG-hierarchy is based on measuring the mind changes of such approximations.
- (J. Miller) A is totally ω-c.a. if for all total $f \leq_T A$, f is ω-c.a.
- The next level of the hierarchy is to consider B such that every function $f \leq_T B$ is computable with the mind-change function provided by the previous level, that is the mind change function is ω-c.a.
- The DG-hierarchy extends this to computable ordinals $< \epsilon_0$. Those with effective Cantor Normal Forms.
- That is, for the second level, it would have the form $\omega k + r$, $k, r \in \omega$. So we count down r many mind changes and then we could move to $\omega(k-1) + r'$, etc.
- For $\alpha < \omega^3$, say, will be given as $\omega^2 k_2 + \omega k_1 + k_0$, with $k_i \in \omega$.

- The basic definition.

Definition (Downey and Greenberg)

A is totally α-c.a. if for all total $f \leq_T A$, f is α-c.a.. That is, there is an approximation f of A, as above, with $|\{s \mid f(x, s+1) \neq f(x, s)\}|$ α-c.a.

- For example, for $\alpha = \omega^2$, we'd replace "g-computable" by g-ω-c.a. so that $g(x) = \lim_s g(x, s)$ with $|\{s \mid g(x, s+1) \neq g(x, s)\}| \leq h(x)$ for some computable h.
- In effect, if $h(x) = 3$, then for x, $f(x)$'s original ordinal would be $\omega \cdot 3 + k_0$. It would get k_0 many mind changes, and then would need to change to $\omega \cdot 2 + k_0'$, etc.
- This hierarchy can be used to classify, unify various constructions and gives rise to a number of new definability results.

Two examples

Theorem (Downey, Greenberg and Weber, 2007)

A c.e. degree **a** *bounds a critical triple (a certain definable configuration) iff* **a** *is not totally* ω*-c.a.*

Theorem

A c.e. degree **a** *bounds a 1-3-1 iff it is not totally* $< \omega^\omega$*-c.a. (This means that for an* $n < \omega$*, there is a* $g \leq_T \mathbf{a}$ *such that* g *is not* ω^n*-c.a.)*

- There is a nice book full of fascinating material on such results, and lots of intersting open questions...

Earlier work

- An earlier notion which has been extensively studied, is the notion of array computability, of Downey, Jockusch and Stob (1989,1996).
- **a** is array computable iff there is a $g \leq_{wtt} \emptyset'$ such that for all $h \leq_T \mathbf{a}$, $h(x)$ can be approximated with at most $g(x)$ many mind changes for almost all x. If **a** is not array computable it is called array noncomputable.
- This is the uniform version of being totally ω-c.a.
- In the same way that there are many characterizations equivalent to being totally ω-c.a., this is true of array noncomputability.

Theorem

The following are equivalent for a a c.e. degree **a**.

(i) **a** *is array noncomputable.*

(ii) **a** *is the degree of a perfect thin* Π_1^0 *class* **(Downey, Jockusch and Stob +Coles, Downey, Herrmann and Jockusch)**.

(iii) **a** *bounds a disjoint pair of c.e. sets with no Turing complete separating set* **(Downey, Jockusch and Stob)**.

(iv) **a** *contains a c.e. set* A *such that* $\exists^\infty n(C(A \upharpoonright n) \geq 2\log n - O(1)$ *where* C *denotes plain Kolmogorov complexity* **(Kummer)**.

(v) **a** *contains a set of packing dimension 1* **(Downey and Greenberg)**.

(vi) *There are left-c.e. reals* $\alpha_0, \alpha_1 \in$ **a** *which have no common upper bound in the cl-degrees of left-c.e. reals* **(Barmpalias, Downey and Greenberg)**.

(vii) *There is a left-c.e. real* $\alpha \in$ **a** *which is not cl-reducible to any random left-c.e. real* **(Barmpalias, Downey and Greenberg)**.

(viii) *There is a set* $A \in$ **a** *which is not cl-reducible to any random left-c.e. real* **(Barmpalias, Downey and Greenberg)**.

The ω-case

Theorem (Downey and Ng, 2018)

There is a c.e. degree $\mathbf{a}$ such that if $\mathbf{a}_1 \vee \mathbf{a}_2 = \mathbf{a}$ then $\mathbf{a}_i$ is not totally ω-c.a. for $i \in \{1, 2\}$.

- Before we sketch the proof, we remark that $\mathbf{a}$ can be chosen to be high_2. This is interesting since the following holds.

Theorem (Downey and Ng, 2018)

If $\mathbf{a}$ is high then any c.e. set of degree $\mathbf{a}$ can be split into a pair of array computable c.e. sets. (Array computable is a strengthening of being totally ω-c.a.)

- Downey and Ng show also that array computable cannot be improved to superlow in this result.

- We meet the requirements:

$$N_e : \text{If } A = \Gamma_e^{W_e \oplus V_e} \text{ and } W_e \oplus V_e = \Delta_e^A, \text{ then one of}$$

$$W_e \text{ or } V_e \text{is not totally } \omega\text{-c.a..}$$

- Drop "e". We e need to ensure that there are *total* functions F^W, G^V computable from W and V respectively, such that one of F^W or G^V is not ω-c.a..
- Effective enumeration $\langle a_i, b_i \rangle$ of all possible ω-c.a. approximations. That is, each $a_i(-,-)$ is a total computable function, and $b_i(-)$ is a partial computable function. We say that a function F is i-approximated if b_i is total and for every x, the number of mind changes on $a_i(x,-)$ is bounded by $b_i(x)$, and $\lim_s a_i(x,s) = f(x)$.
- To make, for example, F^W is not totally ω-c.a., the idea would be to defeat all pairs a_i, b_i, by finding some argument x where we can change $F^W(x)$ more than $b_i(x) = k$ many times; waiting for the approximation $a_i(x,s)$ to agree with the current approximation $F^W(x)[s]$ before each such change. (You "beat it to death".)

- Now, when we define $F^W(x)$ we cannot know $b_i = k$, as the opponent will play this after we define the use of F, at the mother node for the requirement.
- This is important since if we knew the bounds in advance we could have a series of agitators we could put into A in reverse order to kill one of the approximations (showing that, for example, one of V or W is not superlow).
- This problem is overcome by making the construction nonuniform.
- The requirement N will build a pair of functions $\langle F^W, G^V \rangle$ which will be total in the case when the N-hypothesis is true.
- The requirement N is divided into infinitely many subrequirements $N_{i,j}$, and each of these subrequirements $N_{i,j}$ will build a pair of functions $\langle F^W_{i,j}, G^V_{i,j} \rangle$.
- The subrequirement $N_{i,j}$ will itself be divided into infinitely many sub-subrequirements $N_{i,j,k,l}$, which are each aiming to ensure that, if the N-hypothesis is correct, then either F^W is not i-approximated, or G^V is not j-approximated, or $F^W_{i,j}$ is not k-approximated, or $G^V_{i,j}$ is not l-approximated.

- Pick a follower (agitator) x for A, and fresh numbers n_1, n_2, n_3, n_4. Wait for $\gamma(\delta(x))[s]\downarrow$, and then define all of $F^W(n_1), G^V(n_2), F^W_{i,j}(n_3), G^V_{i,j}(n_4)$ convergent with use $\gamma(\delta(x))[s]$. Freeze A. The role of the follower will be to induce changes into A and thus, indirectly, into one of W or V.
- Wait for all of $b_i(n_1), b_j(n_2), b_k(n_3), b_l(n_4)$ to converge. Enumerate x into A and wait for $W \oplus V$-change.
- One of W or V will have changed, and the following is completely symmetric. So suppose that W changed in step 2, so that now both $F^W(n_1)$ and $F^W_{i,j}(n_3)$ are undefined. Now pick $x_1, \cdots, x_{b_i(n_1)}$ ($b_1(n_1)$ many new agitators) such that $x_{m+1} > \gamma(\delta(x_m))$ for all m; define $F^W(n_1)$ on new use $\gamma(\delta(x_{b_i(n_1)}))$. Pick a fresh follower $\tilde{n}_2 > n_2$, and define $G^V(\tilde{n}_2)$ with use $\gamma(\delta(x_{b_i(n_1)}))$. Leave $F^W_{i,j}(n_3)\uparrow$. Increase restraint on A. Note that now, if we put the agitators into A in reverse order, provied that *all* the changes occur in W, we have enough to now kill a_1, b_1 on n_1.

- Wait for $b_j(\tilde{n}_2)$ to converge. Once it converges run the basic beating strategy to try and make F^W not i-approximated at input n_1. That is, we put the followers in in reverse order each time we see a reconvergence of the $a_1(n_1)$ aproximation agreeing with $F^W(n_1)[s]$. This process will succeed unless it is interrupted by a V-change. Go to next step if this happens.
- Now pick fresh followers $y_1, \cdots y_M$ (where $M = b_j(\tilde{n}_2) + b_k(n_3)$) such that $y_{m+1} > \gamma(\delta(y_m))$ for all m; define $F^W_{i,j}(n_3)$ and $G^V(\tilde{n}_2)$ with use $\gamma(\delta(y_{b_j(\tilde{n}_2)}))$. Increase restraint on A. Run the basic beating strategy to make either $F^W_{i,j}$ not k-approximated at input n_3, or G^V not j-approximated at input $\tilde{n}_2$.

Classical Proof

- The original proof of Sacks for $A_1|_T A_2$ gives sets A_i which are totally ω^ω-c.a.
- That is, each total f^{A_i} for each x, we'd need to specify a $d = d(x)$ such that $f^{A_i}(x)$ is ω^d-c.a.
- That is because, in the original proof, each time we get an injury from below, we'd need to reset the use., resulting in an unknown number of injuries, when the opponent re-sets the use.
- This would then force us to choose a new ordinal, and this is multiplicative.
- For example, think about the second requirement. It must defer to the other higher priority requirement, and the amount of deference is determind only when a computation comes about for the higher priority requirement. So this correlates to $\omega + 1$.
- The thrid requirement seems to need $\omega^2 + 1$, etc.

The ω^2 proof

- We are constructing $A_1 \sqcup A_2 = A$, and need to make $\Phi_e^{A_i}(x)$ have a count $\omega k_x + d_x$, and think of this as $R_{e,i,x}$.
- These can be thought of as filtering numbers into A_{i-1}, since the wish to preserve computations.
- Thus there is no conflict between $\langle i, e, x \rangle$ and $\langle i, f, y \rangle$.
- To keep the count down when $\langle i, e, x \rangle$ is injured, for the least x, it grabs all the lower priority $\langle i, e, x' \rangle$ (this is for a single e).
- This "blocking" technique makes the count adhere to being ω^2 for a single e's and without too much trouble gives ω^3.
- Then to make ω^2 a you have to use a nonuniform version of the argument built on a priority tree.

Unbounded type

- We begin with a simpler result:

Theorem

There exist c.e. sets $\varnothing <_T C$ and A, such that if $A = A_1 \sqcup A_2$ is a c.e. splitting of A, and A_1 is ω^2-c.a. then $C \leq_T A_2$.

- Let $\{\langle f_i(\cdot,\cdot), o_i\rangle \mid i \in \omega\}$ list all pairs of primitive recursive functions which are ω^2-c.a.. That is $o_i(x,s)$ lists two numbers $k_s(x), p_s(x) \in \mathbb{N}$ meant to indicate that at stage s the mind change function for f_i is $\omega k_s + p_s$. This approximation obeys the rules that (dropping the "x" when the meaning is clear)
  1. if $f_i(x,s+1) \neq f_i(x,s)$ then $o_i(x,s) > o_i(x,s+1)$. That is, either $k_{s+1} < k_s$ or $p_{s+1} < p_s$, entailing that if $p_s = 0$, then $k_{s+1} < k_s$.
  2. if ever $o_i(x,s) = 0, 0$, then no further changes are allowed to $f_i(x,s)$.
- Note that if a function g is ω^2-c.a. then there is some i with $g(x) = lim_s f_i(x,s)$ with ordinal approximation o_i.
- To prove this we meet

$$R_e : X_e \sqcup Y_e = A \wedge X_e \ \omega^2\text{-c.a.} \ \rightarrow C \leq_T Y_e$$

- Child nodes ρ, which test:

$$R_{e,i} : X_e \sqcup Y_e = A \wedge (\Gamma^{X_e} = \lim_s f_i \text{ with ordinal } o_i) \rightarrow \exists \Delta^{Y_e} = C.$$

  A ρ-node has outcomes $\infty <_L f$.
- Of course in the background:

$$P_e : \overline{C} \neq W_e.$$

  These are met as usual, pick a follower x wait till $x \in W_{e,s}$ (i.e. realized) and put x into C. The key problem is how to achieve these goals whilst still meeting the R_e's.
- We consider the situation of a single σ-node below $\rho^\frown\infty$, trying to get a follower into A after it is realized.
- σ indicates that it wants a follower. At the first $\tau^\frown\infty$ stage s after this flag, we will appoint an *anchor* z and target this for σ. We define $\Gamma(z, s)$ to be large, say s_0.
- We wait for a stage where $\ell(\rho, s_1) > z$, and then ρ will have declared its count $k_0(z), p_0(z) = k, p$. Thus $f_i(z) = \Gamma^{X_\tau}(z) = s_0[s_1]$ is henceforth permitted at most $\omega k_0 + p_0$ many mind changes.

- At the next σ-stage we will appoint a fresh number x_0 to follow P_σ. Note that $x_0 > \gamma(z, s)$. At the next $\rho^\frown\infty$-stage s_1, stage we will define $\Delta^{Y_\tau}(x_0) = 0$ with use s_1.

- Suppose that x_0 is realized at stage s_2.
- Now at the first σ-stage after s_2, we will put $n = \gamma(z, s)$ into A. It must enter X_e or Y_e.
- If n enters Y_e we can put x_0 into C meeting P_σ since Y_e can comprehend the entry.
- It n enters X_e, then (at the next $\tau\widehat{\ }\infty$-stage s_4) we will re-define $\Gamma^{X_e}(z) = s_4$, say, and lift $\gamma^{X_e}(z, s_4 + 1)$ to a fresh number (such as s_4) and at the next σ-stage, pick a new follower $x_1 > \gamma(x, s_4 + 1)$ and begin to repeat the cycle.

- Key point: We will only visit σ at the next $\rho^\frown\infty$-stage s_5, where it must be that $f_i(z, s_5) = \gamma^{X_e}(s, s_4 + 1)$, and hence $o_i(z)[s_5]$ has decreased.
- We repeat this cycle at most $\omega k_0 + p_0$ many times, since any more than that, f_ρ, o_ρ are not the correct functions witnessing that Γ^{X_e} is ω^2-c.a..
- Dealing with two mothers involves a complicated combinatorial argument, but is okay in the end. (The final write up does the argument somewhat differently, but the timing difficulties are the same.)
- After the fact, you realize that the construction only really needs that we know the level below ϵ_0. This gives the full theorem, and then you observe that it actually works for degrees using "agitated intervals".

Questions

- Clearly lots of nice questions:
- Relationships of converse computability theory with the hierarchy.
- (conjecture, likely true), you can combine the construction of a degree that does not bound a Slaman triple with one that is not the join of totally ω-c.a. c.e. degrees. This is a elementarily definable class which consists of only high$_2$ non-low$_2$ degrees. This likely also works for contiguous degrees.
- Is there totally ω c.a. computably enumerable degree with no maximal totally ω^n-c.a. degrees above it?
- A new hierarchy was introduced by Ambos-Spies, Downey, Monath where we looked at mind changes for wtt-functionals. This allowed for classification of the sets *wtt*-reducible to maximal ones. It is fundamentally unexplored.

Thank You

- (Downey and Greenberg) *A Hierarchy of Turing Degrees: A Transfinite Hierarchy of Lowness Notions in the Computably Enumerable Degrees, Unifying Classes and Natural Definability*, (with Noam Greenberg), Annals of Mathematics Studies, Vol. 206 Princeton University Press
- Splitting into degrees of low computational strength, (Downey and Ng), APAL, 2018, Vol. 169, 803-834.
- Notes on Sacks Splitting Theorem, (Ambos-Spies, Downey, Monath and Ng), in preparation.
- A Hierarchy of Computably Enumerable Degrees II: Some Recent Developments and New Directions, with Noam Greenberg and Ellen Hammatt, *New Zealand Journal of Mathematics, Vaughan Jones Special Issue.* Vol. 52 (2021), 175-231.