Extensions of embeddings in the Σ^0_2 enumeration degrees

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Beyond the Turing degrees: The enumeration degrees

Definition (various authors, 1950s)

For $A, B \subseteq \mathbb{N}$, we say that A is enumeration reducible to B ($A \leq_{\mathrm{e}} B$) if

every enumeration of B computes an enumeration of A.

Example

Let B be a maximal independent set of vertices in a computable graph. Then $B^c \leq_{\mathrm{e}} B$ (but $B \not\leq_{\mathrm{e}} B^c$ in general.)

The enumeration degrees (e-degrees) are defined from $\leq_{\mathrm{e}}$ in the same way that the Turing degrees are defined from $\leq_{\mathrm{T}}$.

They form an upper-semilattice under $\leq_{\mathrm{e}}$ and the usual effective join.

Beyond the Turing degrees: The enumeration degrees

The Turing degrees embed into the e-degrees in a natural way:

$$A \leq_{\mathrm{T}} B \quad \text{if and only if} \quad A \oplus A^c \leq_{\mathrm{e}} B \oplus B^c$$

so the map $A \mapsto A \oplus A^c$ induces an embedding.

Questions we can ask about a partial order

1. Is it linear?
2. Which finite partial orders embed into it?
3. Is it dense?
4. Given finite partial orders $\mathcal{P} \subseteq \mathcal{Q}$, can every embedding of $\mathcal{P}$ into it be extended to an embedding of $\mathcal{Q}$?
5. (!) Given a first-order sentence in the language of $\{\leq\}$, can we algorithmically decide if it is true?

For many degree structures, the answer to 5 is very much no (Slaman, Woodin 1997).

A central question

On the other hand, every finite partial order embeds into the e-degrees (corollary of Sacks 1963), so we can compute if a sentence of the form

$$\exists a_0 \exists a_1 \cdots \exists a_n \text{ (Boolean combination of } a_i \leq a_j)$$

holds by checking whether the Boolean combination is consistent with the axioms of partial orders.

At what point does computability break down?

A countable substructure: The Σ^0_2 e-degrees

We are working on this question for the Σ^0_2 e-degrees.

Reasons to study the Σ^0_2 e-degrees:

1. They are analogous to the c.e. Turing degrees
2. Any nontrivial automorphism of the e-degrees must move some Σ^0_2 e-degree (Slaman, Soskova 2017)
3. They exhibit unusual order-theoretic phenomenon (next slide)

The Σ_2^0 e-degrees: Ahmad pairs

The Σ_2^0 e-degrees are dense (Cooper).

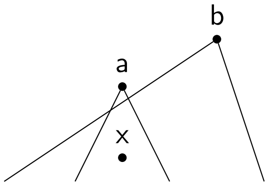
Compare: The c.e. Turing degrees are dense (Sacks density).

Question (Cooper)

Does the Σ_2^0 e-degrees satisfy the same first-order sentences as the c.e. Turing degrees?

Theorem (Ahmad 1989)

In the Σ_2^0 e-degrees, there are incomparable **a** and **b** such that if $\mathbf{x} <_{\mathrm{e}} \mathbf{a}$, then $\mathbf{x} \leq_{\mathrm{e}} \mathbf{b}$.



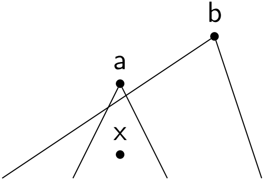
Notice that **a** cannot be the join of two degrees below it.

On the other hand, in the c.e. Turing degrees, every degree is the join of two degrees below it (Sacks splitting).

There are no Ahmad triples

Theorem (Ahmad 1989)

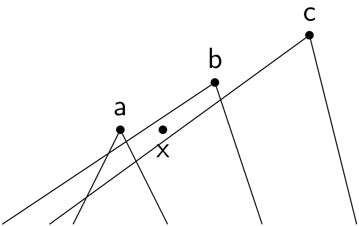
There are incomparable **a** and **b** such that if $\mathbf{x} <_{\mathrm{e}} \mathbf{a}$, then $\mathbf{x} \leq_{\mathrm{e}} \mathbf{b}$.



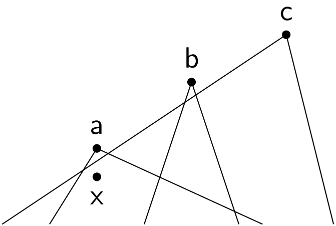
Theorem (G., Lempp, Ng, Soskova 2021)

For every incomparable **a**, **b**, **c**, there is some **x** such that:

- $\mathbf{x} <_{\mathrm{e}} \mathbf{a}$ but $\mathbf{x} \not\leq_{\mathrm{e}} \mathbf{b}$, OR
- $\mathbf{x} <_{\mathrm{e}} \mathbf{b}$ but $\mathbf{x} \not\leq_{\mathrm{e}} \mathbf{c}$.



(Actually we just need $\mathbf{a} \not\leq \mathbf{b}$, so **a** and **c** could be comparable or even equal.)



Reformulating as extensions of embeddings

Theorem (Ahmad 1989)

There are incomparable $\mathbf{a}$ and $\mathbf{b}$ such that if $\mathbf{x} <_{\mathrm{e}} \mathbf{a}$, then $\mathbf{x} \leq_{\mathrm{e}} \mathbf{b}$.

Reformulation:

> *Not every embedding of the antichain $\{a, b\}$ (into the Σ^0_2 e-degrees) can be extended to an embedding of $\{a, b, x\}$ where $x < a$ and $x \not\leq b$.*

Our result can be reformulated similarly:

> *Every embedding of the antichain $\{a, b, c\}$ can be extended to one of the following:*
> - *an embedding of $\{a, b, c, x\}$ where $x < a$ and $x \not\leq b$*
> - *an embedding of $\{a, b, c, x\}$ where $x < b$ and $x \not\leq c$.*

(Actually, there are four choices here because in the first ordering, we didn't specify the relationship between x and c, and in the second ordering, we didn't specify the relationship between x and a.)

“Disjunctive” results

Our result on no “Ahmad triples” generalizes the following

Theorem (Ahmad 1989)

There are no “symmetric Ahmad pairs”, i.e., for every incomparable **a** and **b**, there is some **x** such that:

- $\mathbf{x} <_{\mathrm{e}} \mathbf{a}$ but $\mathbf{x} \not\leq_{\mathrm{e}} \mathbf{b}$, or
- $\mathbf{x} <_{\mathrm{e}} \mathbf{b}$ but $\mathbf{x} \not\leq_{\mathrm{e}} \mathbf{a}$.

In other words, every embedding of the antichain $\{a, b\}$ can be extended to one of the following:

- an embedding of $\{a, b, x\}$ where $x < a$ and $x \not\leq b$
- an embedding of $\{a, b, x\}$ where $x < b$ and $x \not\leq a$.

What's the goal here?

We'd like to give an algorithm for deciding the **two-quantifier** first-order theory of the Σ^0_2 e-degrees. In terms of quantifier alternations, two quantifiers is the most we can hope to decide (Kent 2006).

Fact: Every two-quantifier sentence can be thought of as a disjunctive extension of embeddings problem.

Our focus (for now) is the following special case:

One-point extensions of antichains problem

Given a (finite) **antichain** $\mathcal{P}$ and **one-point** extensions $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$ of $\mathcal{P}$, decide whether every embedding of $\mathcal{P}$ into the Σ^0_2 e-degrees extends to an embedding of some $\mathcal{Q}_i$.

$n = 0$ was solved by Lempp, Slaman, Sorbi 2005. $n \geq 1$ is open.

One-point extensions of antichains problem

We'll restrict ourselves further to one-point extensions where the new element is **not above** any of the old elements.

Given an antichain $\mathcal{P} = \{a_0, \ldots, a_k\}$ and such one-point extensions $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$ of $\mathcal{P}$, how might we embed $\mathcal{P}$ in a way which does not extend to an embedding of any $\mathcal{Q}_i$?

Let x_i denote the new element added by $\mathcal{Q}_i$.

Say $x_i < a_j$ in $\mathcal{Q}_i$. By density, given any $\mathbf{a}_j$, we can find some (nonzero) $\mathbf{x}_i <_{\mathrm{e}} \mathbf{a}_j$.

Whether this defines an embedding of $\mathcal{Q}_i$ depends on the set of $j' \neq j$ for which $\mathbf{x}_i <_{\mathrm{e}} \mathbf{a}_{j'}$.

One-point extensions which put a new element below exactly one old element

Suppose now that in $\mathcal{Q}_i$, we have $x_i < a_j$ and $x_i \mid a_{j'}$ for $j' \neq j$. We shall call such $\mathcal{Q}_i$ singleton extensions.

(If no $\mathcal{Q}_i$ is a singleton extension, we can make $\mathbf{a}_0, \ldots, \mathbf{a}_k$ pairwise minimal pairs to prevent extensions to every $\mathcal{Q}_i$.)

If $\mathbf{a}_0, \ldots, \mathbf{a}_k$ is an antichain which does not extend to an embedding of $\mathcal{Q}_i$ then:

> *Every $\mathbf{x} <_{\mathrm{e}} \mathbf{a}_j$ must be below some other $\mathbf{a}_{j'}$.*

If $\mathbf{a}_j$ has the above property we say that it is a weak Ahmad base.

- If **a** and **b** form an Ahmad pair, then **a** is a weak Ahmad base.
- Not all weak Ahmad bases come from Ahmad pairs (G., Lempp, Ng, Soskova 2021).

We want to understand which degrees can be weak Ahmad bases, because in the general (i.e., disjunctive) situation, we want to construct an embedding $\mathcal{P}$ which cannot be extended to any $\mathcal{Q}_i$ in a finite list, not just a single $\mathcal{Q}_i$.

Theorem (G., Lempp, Ng, Soskova ongoing)

If **a** and **b** form an Ahmad pair, then **b** is not a weak Ahmad base.

This generalizes our result on "no Ahmad triple".

In fact, we believe we have proved the following stronger statement (∗):

> *If $\mathbf{a}_0, \ldots, \mathbf{a}_j$ are incomparable weak Ahmad bases, then for each i, there is some $\mathbf{x}_i <_{\mathrm{e}} \mathbf{a}_i$ such that $\mathbf{x}_i \not\leq_{\mathrm{e}} \mathbf{a}_{i'}$ for all $i' \neq i$.*

(To derive the theorem from (∗), take $j = 1$, $\mathbf{a}_0 = \mathbf{a}$, $\mathbf{a}_1 = \mathbf{b}$.)

A necessary condition for embeddings which fail to extend

Suppose $\mathbf{a}_0, \ldots, \mathbf{a}_k$ is an antichain which does not extend to an embedding of any of $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$.

WLOG we can put all singleton extensions at the front. Fix $j \leq k$ such that the $\mathcal{Q}_i$ for $i \leq j$ are those whose new element x_i lies below a_i and no other $a_{i'}$.

Then $\mathbf{a}_0, \ldots, \mathbf{a}_j$ are weak Ahmad bases, so (assuming $(*)$) for each $i \leq j$, we can choose some $\mathbf{x}_i <_{\mathrm{e}} \mathbf{a}_i$ such that $\mathbf{x}_i \not\leq_{\mathrm{e}} \mathbf{a}_{i'}$ for other $i' \leq j$.

Each $\mathbf{x}_i$ must be below some $\mathbf{a}_{j'}$, where $j' > j$ (otherwise we could extend to $\mathcal{Q}_i$ using $\mathbf{x}_i$).

Furthermore, there cannot be any $\mathcal{Q}_m$ such that for $j' \leq k$,

$$\mathbf{x}_i \leq_{\mathrm{e}} \mathbf{a}_{j'} \quad \text{if and only if} \quad x_m \leq a_{j'} \text{ in } \mathcal{Q}_m.$$

Lemma (assuming ($*$))

If there is an embedding of $\{a_0, \ldots, a_k\}$ which does not extend to an embedding of any of $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$, then there is a function ν, defined on the set of i such that $\mathcal{Q}_i$ is a singleton extension, with the following properties for each $i \in \mathrm{dom}(\nu)$:

I. *$\nu(i)$ is a nonempty subset of $\{0, \ldots, k\} - \mathrm{dom}(\nu)$*

II. *There is no $\mathcal{Q}_m$ such that*

$$\{i\} \cup \nu(i) = \{j' \leq k : x_m \leq a_{j'} \text{ in } \mathcal{Q}_m\}.$$

Example

Every embedding of the antichain $\{a_0, a_1\}$ extends to an embedding of one of the following:

- $\mathcal{Q}_0$ adds x_0 where $x_0 < a_0$ only ← singleton extension
- $\mathcal{Q}_1$ adds x_1 where $x_1 < a_0, a_1$

because (by I) the only possibility for $\nu(0)$ is $\{1\}$, but then II is violated because of $\mathcal{Q}_1$.

A further necessary condition

Recalling the setup from two slides ago:

- $\mathbf{a}_0, \ldots, \mathbf{a}_k$ does not extend to an embedding of any of $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$.
- The $\mathcal{Q}_i$ for $i \leq j$ are those whose new element x_i lies below a_i and no other $a_{i'}$.
- For each $i \leq j$, we can choose some $\mathbf{x}_i <_{\mathrm{e}} \mathbf{a}_i$ such that $\mathbf{x}_i \not\leq_{\mathrm{e}} \mathbf{a}_{i'}$ for other $i' \leq j$.

Consider the join $\mathbf{x}_i \vee \mathbf{x}_{i'}$, for $i < i' \leq j$.

The set of $j' \leq k$ such that $\mathbf{x}_i \vee \mathbf{x}_{i'} \leq_{\mathrm{e}} \mathbf{a}_{j'}$ is exactly the intersection of that for $\mathbf{x}_i$ and $\mathbf{x}_{i'}$.

This intersection cannot be "realized" by any $\mathcal{Q}_m$ either.

We may apply a similar analysis to any join of the $\mathbf{x}_i$.

A working conjecture

We believe that the previous necessary conditions are sufficient, i.e.,

> *As long as there is some function ν with the required properties, then we can construct an embedding of the antichain $\{a_0, \ldots, a_k\}$ which fails to extend.*

If true, this would give us an algorithm for deciding whether every embedding of a given finite antichain $\mathcal{P}$ extends to an embedding of some $\mathcal{Q}_i$, where $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$ are one-point extensions of $\mathcal{P}$ where the new element is not above any old elements.

Thanks!