Computability Theory, Set Theory and Geometric Measure Theory

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Abstract

We will explore issues within Geometric Measure Theory using methods from Computability Theory and Set Theory. Except when stated otherwise, this is joint work with Jan Reimann.

Ingredients from Logic

- Lightface characterization of co-analytic sets: A is co-analytic iff there is a $p \in 2^\omega$ such that $x \in A$ is uniformly Σ_1-definable relative to p in $L_{\omega_1^{x p}}(x, p)$.
  - If $V = L$, then $\{x : x \in L_{\omega_1^x}\}$ can be used to define co-analytic sets by transfinite recursion.
- Set theory of the continuum.
  - Especially the understanding of measure in generic extensions.
- Semi-recently uncovered connections between algorithmic randomness and Hausdorff dimension.
- Determinacy.
  - We will not explore this topic here, but recent results of Crone, Fishman and Jackson (2020) show that some of the regularity properties we will cite for analytic sets apply more generally under determinacy assumptions.

Gauge Functions and General Hausdorff Dimension

Definition

A *gauge function* is a function $h : (0, \infty) \to (0, \infty)$ which has the following properties:

- continuous
- increasing
- $\lim_{t \to 0^+} h(t) = 0$

Example

$h(t) = t^s$, for $s > 0$.

Gauge Functions and General Hausdorff Dimension

Definition

Let h be a gauge function. For a set $A \subseteq 2^\omega$ (or ω^ω, $\mathbb{R}^n$ etc.), define

$$H^h(A) = \lim_{\delta \to 0} \inf_{\substack{A \subseteq \cup F_i \\ \max d(F_i) < \delta}} \sum_{i=1}^{\infty} h(d(F_i))$$

where $\{F_i\}$ is a sequence of closed (open) sets covering A and $d(F_i)$ is the diameter of F_i.

- When $h(t)$ is t^s, $H^h = H^s$ is the usual s-dimensional Hausdorff outer measure.
- Gauge functions provide a more finely graded calibration of measure and thereby of dimension than is given by the usual family $\{t \mapsto t^s : s \in [0, 1]\}$.

Gauge Functions and General Hausdorff Dimension

Definition

The Hausdorff dimension of a set A is the number d such that whenever $d_0 < d < d_1$, $H^{d_0}(A) = \infty$ and $H^{d_1} = 0$.

Example

The Cantor middle-third set, which has dimension $\log(2)/\log(3)$ is null with respect to linear (Lebesgue) measure.

Point-to-Set Principle

Definition (Lutz, Mayordomo 2002)

For any $x \in 2^\omega$,

$$\dim_{\mathrm{H}}^{\mathit{eff}}(x) = \liminf_{\ell \to \infty} \frac{K(x \upharpoonright \ell)}{\ell}$$

where K denotes Kolmogorov complexity.

Theorem (J. Lutz and N. Lutz 2017)

For $A \subseteq 2^\omega$, the Hausdorff dimension of a set A is equal to
the infimum over all $B \subseteq \mathbb{N}$
of the supremum over all $x \in A$
of $\dim_{\mathrm{H}}^{\mathit{eff}(B)}(x)$, the effective-relative-to-B H-dimension of x.

Gauge Functions and General Hausdorff Dimension

The clarity of Hausdorff dimension transfers only partially to gauge measures.

Recall:

Definition

Write $h \prec g$ to indicate that $\lim_{t \to 0^+} \frac{g(t)}{h(t)} = 0$.

Similarities:

- If $H^h(A)$ is finite and $h \prec g$ then $H^g(A) = 0$.
- If $H^h(A)$ is not zero and $j \prec h$ then $H^j(A)$ is infinite.

Difference:

- (Besicovitch 1956) If $H^h(A) = 0$ then there is a j with $j \prec h$ such that $H^j(A) = 0$.

Sets of non-σ-finite measure

Definition

A set A is σ-finite for H^h iff A is a countable union of sets A_i, such that each $H^h(A_i)$ is finite.

Improved observation from previous slide:

- If $H^h(A)$ is not zero and $j \prec h$ then A is non-σ-finite for H^j.

Example

For d in $(0,1)$, the set $D_d^{eff} = \{x : \dim_{\mathrm{H}}^{eff}(x) = d\}$ has Hausdorff dimension d and is non-σ-finite for H^d.

Question

Is there a useful point-to-set formulation of a set's being non-σ-finite for H^h?

Sets of Strong Dimension h

Definition

A set E has *strong dimension h* iff

$$\forall f[f \prec h \Rightarrow H^f(E) = \infty]$$

$$\forall g[h \prec g \Rightarrow H^g(E) = 0]$$

As a limiting case, E has strong dimension 0 iff for all g, $H^g(E) = 0$.

Example

A line segment within the plane has strong dimension 1.

Sets of Strong Dimension h

Theorem (Besicovitch 1956, generalized Rogers 1962)

If E is compact and is non-σ-finite for H^h, then there is a g such that $h \prec g$ and E is non-σ-finite for H^g.

Thus, if E is compact then E cannot have strong dimension h and be non-σ-finite for H^h.

Theorem (Davies 1956 for x^s, Sion and Sjerve 1962)

If E is analytic and is non-σ-finite for H^h, then there is a compact subset of E that is non-σ-finite for H^h.

Hence, we can make the above observation for analytic sets.

It would be interesting to find proofs of these theorems using effective methods.

Sets of Strong Dimension h

Theorem (Besicovitch 1963)

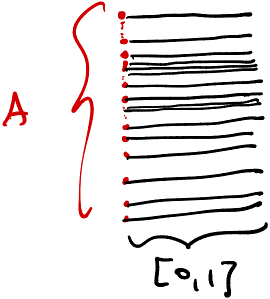
If CH then there is a set $E \subset \mathbb{R}^2$ such that E has strong linear dimension and is non-σ-finite for linear measure.

Theorem (Combining Besicovitch 1963 with Erdős, Kunen and Mauldin 1981)

If $V = L$ there there is a Π^1_1 set $E \subseteq \mathbb{R}^2$ such that E has strong linear dimension and is non-σ-finite for linear measure.

Proof Sketch

$E = A \times [0, 1]$, where A is a small uncountable set.



- (Besicovitch) CH implies that there is an uncountable set A such that any open cover of $\mathbb{Q}$ is an open cover of A.
- (Erdős, Kunen and Mauldin) $V = L$ implies that there is a co-analytic example of such an A.

Other Examples under $V = L$

capacitability

Theorem (Besicovitch and Davies (independently) 1952)

If A is an analytic subset of 2^ω and $\dim_{\mathrm{H}}(A) = d$, then for every $s < d$ there is a closed set $C_s \subseteq A$ such that $s \leq \dim_H(C) \leq d$.

Theorem (Slaman)

If $V = L$ then the maximal thin Π^1_1 set, $\{x : x \in L_{\omega_1^x}\}$, has Hausdorff dimension 1, but all of its closed subsets are countable.

Other Examples under $V = L$

projections of sets of positive dimension

Theorem (Marstrand 1952)

Let $E \subseteq \mathbb{R}^2$ be analytic. Then, for almost every angle $\theta \in [0, 2\pi]$,

$$\dim_H(p_\theta\, E) = \min\{\dim_H(E), 1\},$$

where $p_\theta(x, y) = x \cos\theta + y \sin\theta$. Moreover, if $\dim_H(E) > 1$, then $\mu(p_\theta\, E) > 0$, for almost every angle θ.

Theorem (Slaman and Stull (in progress))

If $V = L$ then for every $s \in (0, 1)$ there is a $\underset{\sim}{\Pi}^1_1$ set $E \subseteq \mathbb{R}^2$ such that $\dim_H(E) = 1 + s$ but

$$\dim_H(p_\theta E) = s$$

for every $\theta \in (0, 2\pi)$.

Borel Conjecture

return to discussion of strong dimension h

Definition

A set $E \subseteq \mathbb{R}$ has *strong measure 0* iff for any sequence of positive real numbers $\{\epsilon_i\}$ there is a sequence of open intervals $\{O_i\}$ such that for each i, O_i has length ϵ_i, and $E \subseteq \cup_{i=1}^{\infty} O_i$.

Borel (1919) conjectured that strong measure 0 implies countable (BC).

Theorem

- *(Sierpiński 1928) CH implies that there is an uncountable set of strong measure 0.*
- *(Laver 1976) $Con(ZFC)$ implies $Con(ZFC + BC)$.*

Borel Conjecture

Theorem (Besicovitch 1955)

A set E has strong dimension 0 iff it has strong measure 0.

Theorem (Another variation on Besicovitch 1963)

$\neg BC$ implies that there is a subset of $\mathbb{R}^2$ which has strong linear dimension and which is non-σ-finite for linear measure.

A Challenge

Question

Does the Borel Conjecture imply that there do not exist h and A such that A has strong dimension h and A is not σ-finite for H^h?

The conceptual challenge is to overcome the intractability of the property that A is non-σ-finite for H^h.

Understanding Non-σ-finiteness

A case study

Consider Π^0_1 subsets of $2^\omega \times 2^\omega$ and linear measure H^1.

Exercise

The set of indices for Π^0_1 subsets C of $2^\omega \times 2^\omega$ such that $H^1(C) \neq 0$ is arithmetic.

By the compactness of $2^\omega \times 2^\omega$, we can assume that all the open covers in the definition of $H^1(C)$ are finite, which means that the prima facie definition of "$H^1(C) \neq 0$" can be expressed arithmetically.

Understanding Non-σ-finiteness

Definition

Let $N\sigma Finite$ be the of indices for Π^0_1 subsets C of $2^\omega \times 2^\omega$ such that C is non-σ-finite for H^1

Theorem

$N\sigma Finite$ is Σ^1_1-complete.

Here is a more familiar situation which is analogous.

Exercise

The set of indices for Π^0_1 subsets C of 2^ω such that C is uncountable Σ^1_1-complete.

Use Cantor's theorem: C is uncountable iff C has a perfect subset.

Understanding Non-σ-finiteness

NσFinite is Σ^1_1

The ingredients in the proof of Davies's (1956) theorem about capacitability of non-σ-finiteness entail the following:

> C is non-σ-finite for H^1 iff there is perfect tree of closed sets such that each path corresponds to a closed set of H^1-positive measure.

- It follows that $N\sigma Finite$ is a Σ^1_1 set.
- Show that $N\sigma Finite$ is Σ^1_1-hard by direct construction, analogous to the analysis of Cantor's theorem.

Remark

We also exhibit a perfect tree of pairwise-disjoint closed subsets of positive measure in the example that D^{eff}_d is non-σ-finite for H^d.

The End