Three views of LPO and LLPO

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LPO: Limited Principle of Omniscience

LPO: If $f : \mathbb{N} \to \{0, 1\}$ then 0 is in its range or it isn't.

Written as an $\forall\exists$ formula:

$$\forall f\, \exists n\, (n = 0 \vee n = 1 \wedge (n = 0 \leftrightarrow \exists t\, (f(t) = 0)))$$

A (the) realizer for LPO:

$$R_{\text{LPO}}(f) = \begin{cases} 0 & \text{if } 0 \in \text{Range}(f) \\ 1 & \text{if } 0 \notin \text{Range}(f) \end{cases}$$

LLPO: Lesser Limited Principle of Omniscience

LLPO: If $f : \mathbb{N} \to \{0, 1\}$ and $0 \in \mathrm{Range}(f)$, then the first 0 occurs at an even integer or at an odd integer.

Written as an $\forall\exists$ formula:

$$\forall f \, \exists n \, (\exists t \, (f(t) = 0) \to n \equiv_{\text{mod } 2} \mu t(f(t) = 0))$$

Values of a realizer:

n	0	1	2	3	4	...
$f(n)$	1	1	0	1	0	...

$R_{\text{LLPO}}(f) = 0$

n	0	1	2	3	4	...
$g(n)$	1	1	1	0	0	...

$R_{\text{LLPO}}(g) = 1$

n	0	1	2	3	4	...
$h(n)$	1	1	1	1	1	...

$R_{\text{LLPO}}(h) = 0 \text{ or } 1$

Weihrauch reductions

Suppose P and Q are problems.

We say P is Weihrauch reducible to Q (and write P $\leqslant_W$ Q) if there are (partial) computable functionals Φ and Ψ such that if *p* is an instance of P, then

- $\Phi(p)$ is an instance of Q and
- for any solution *s* of $\Phi(p)$, $\Psi(s, p)$ is a solution of *p*.

That is, if R_Q is any realizer of Q, then $\Psi(R_Q(\Phi(p)), p)$ is a realizer for P.

If Ψ does not use the original problem *p*, we say P is strongly Weihrauch reducible to Q.

Weihrauch reductions

If R_Q is any realizer of Q, then $\Psi(R_Q(\Phi(p)), p)$ is a realizer for P.

- Φ is a pre-processor, turning P problems into Q problems.
- Ψ is a post-processor, turning Q solutions (with copies of P problems) into P solutions.
- Φ and Ψ uniformly turn any Q realizer into a P realizer.

Example of a Weihrauch reduction

We will show that LLPO $\leqslant_W$ LPO, relying on post-processing.

Instructions for Φ:

Input an LLPO problem p.

Do nothing and output p.

Instructions for Ψ:

Input p and $R_{\text{LPO}}(\Phi(\text{p}))$.

Note: $R_{\text{LPO}}(\Phi(\text{p}))$ is the LPO solution for p.

If $R_{\text{LPO}}(\text{p}) = 1$ (so p is all 1s)

then output 1 and halt.

If $R_{\text{LPO}}(\text{p}) = 0$ (so p contains a 0)

then find the first 0 and output the location (mod 2).

Ψ is partial. A false value for $R_{\text{LPO}}(\Phi(\text{p}))$ may loop.

Example of a Weihrauch reduction

We will show that LLPO $\leqslant_W$ LPO, relying on pre-processing.

Instructions for Φ:

Input an LLPO problem p.

For each n, define q(n) by

if there is an $m \leqslant n$ such that $\mathrm{p}(m) = 0$ and the first such m is even, then set $\mathrm{q}(n) = 0$, and

set $\mathrm{q}(n) = 1$ otherwise.

Instructions for Ψ:

Input p and $R_{\mathrm{LPO}}(\Phi(\mathrm{p}))$.

Output $R_{\mathrm{LPO}}(\Phi(\mathrm{p}))$.

Φ and Ψ are total. Ψ doesn't use p, so LLPO $\leqslant_{sW}$ LPO.

Parallelization and Weihrauch reducibility

For a Weihrauch problem P, the parallelization is denoted by $\widehat{\mathrm{P}}$.

$\widehat{\mathrm{P}}$ accepts a sequence of P problems as input, and outputs the sequence of their solutions.

For example, written as an $\forall\exists$ formula, $\widehat{\mathrm{LPO}}$ is

$$\forall \langle f_i \rangle \, \exists \langle n_i \rangle \, \forall i \, (n_i = 0 \lor n_i = 1 \land (n_i = 0 \leftrightarrow \exists t \, (f_i(t) = 0)))$$

which is a set existence statement.

Reverse mathematics: The base theory

Reverse Mathematics measures the strength of theorems by proving equivalence results over...

The base theory RCA_0:

Variables for natural numbers and sets of natural numbers

Axioms

Arithmetic axioms
(e.g. $n + 0 = n$ and $n + (m') = (n + m)'$)

Induction for particularly simple formulas

Recursive comprehension:
If you can compute a set, then it exists.

Reverse mathematics: The big five

Many theorems of mathematics are equivalent to one of four statements (over the base theory RCA_0). [3]

$$\mathsf{RCA}_0 < \mathsf{WKL}_0 < \mathsf{ACA}_0 < \mathsf{ATR}_0 < \Pi^1_1\text{-}\mathsf{CA}_0$$

Where does $\widehat{\mathsf{LPO}}$ fit?

Prop (RCA_0): The following are equivalent:

(1) ACA_0.

(2) $\widehat{\mathsf{LPO}}$.

For the reversal, find the range of an arbitrary injection on $\mathbb{N}$.

Reverse mathematics: The big five

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Where does $\widehat{\mathsf{LLPO}}$ fit?

Prop (RCA_0): The following are equivalent:

(1) WKL_0.

(2) $\widehat{\mathsf{LLPO}}$.

For the reversal, separate the ranges of two injections with disjoint ranges.

Computability theoretic observations

There is a computable injection on $\mathbb{N}$ with a range that computes $0'$.

There is a computable $\widehat{\mathrm{LPO}}$ problem such that every solution computes $0'$.

There is an ω model of WKL_0 containing only low sets.

Every computable $\widehat{\mathrm{LLPO}}$ problem has a low solution.

Thus, $\widehat{\mathrm{LPO}} \not\leq_{\mathrm{W}} \widehat{\mathrm{LLPO}}$, and so $\mathrm{LPO} \not\leq_{\mathrm{W}} \mathrm{LLPO}$.

Higher order reverse mathematics

Work in collaboration with Carl Mummert.

Kohlenbach [5] proposed an extension of the axiom systems of reverse mathematics to all finite types.

In this setting, we can prove equivalences between Skolemized functional existence statements.

As an example,

$$\text{LPO} : \forall f \, \exists n \, (n = 0 \vee n = 1 \wedge (n = 0 \leftrightarrow \exists t \, (f(t) = 0)))$$

$$(\text{LPO}) : \exists R_{\text{LPO}} \, \forall f (R_{\text{LPO}}(f) = 0 \vee R_{\text{LPO}}(f) = 1 \wedge \\ (R_{\text{LPO}}(f) = 0 \leftrightarrow \exists t \, (f(t) = 0)))$$

Higher order RM: The base theory

Kohlenbach's [5] RCA_0^ω includes functionals of higher type, like $f : 2^{\mathbb{N}} \to \mathbb{N}$ and $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

It includes:

- Restricted induction
- Primitive recursion (on $\mathbb{N}$ with parameters)
- λ-abstraction

Naïvely, if you can compute a functional, it exists.

(LPO) as an ACA_0 analog

$$(\mathrm{LPO}): \exists R_{\mathrm{LPO}} \forall f (R_{\mathrm{LPO}}(f) = 0 \vee R_{\mathrm{LPO}}(f) = 1 \wedge (R_{\mathrm{LPO}}(f) = 0 \leftrightarrow \exists t\, (f(t) = 0)))$$

(LPO) is Kohlenbach's $(\exists^2)$. The functional R_{LPO} is Kleene's E^2.

$\mathrm{RCA}_0^\omega + (\mathrm{LPO})$ is a conservative extension of ACA_0 for Π^1_2 formulas.[6]

And for (LLPO), there is a surprise:

Prop: (RCA_0^ω) The following are equivalent:

(1) (LPO)

(2) (LLPO)

$\mathsf{RCA}_0^\omega \vdash (\mathsf{LLPO}) \to (\mathsf{LPO})$

Proof sketch: Working in RCA_0^ω, suppose we have R_{LLPO}.

Let $f = \langle 1, 1, 1, \dots \rangle$, and suppose $R_{\mathrm{LLPO}}(f) = 0$.

Define a sequence of inputs:

$$g_n(m) = \begin{cases} 1 & \text{if } m \neq 2n+1 \\ 0 & \text{if } m = 2n+1 \end{cases}$$

For every n, $R_{\mathrm{LLPO}}(g_n) = 1$, so

$$\lim_n R_{\mathrm{LLPO}}(g_n) = 1 \neq 0 = R_{\mathrm{LLPO}}(f) = R_{\mathrm{LLPO}}(\lim_n g_n)$$

and R_{LLPO} is effectively sequentially discontinuous.

Apply Prop. 3.7 of Kohlenbach [5] to obtain (LPO).

If $R_{\mathrm{LLPO}}(f) = 1$, revise the definition of g_n.

Grilliot's trick and Kohlenbach's proposition

We can replace the use of Prop. 3.7 of Kohlenbach [5] with part of the proof of Lemma 1 of Grilliot [4].

Let f and g_n be as before. Define the functional $J : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ for $h : \mathbb{N} \to 2$ and $j \in \mathbb{N}$ by

$$J(h)(j) = \begin{cases} 1 & \text{if } \forall x \leqslant j\,(h(x) \neq 0), \\ g_i(j) & \text{if } i \leqslant j \wedge i = \mu t(h(t) = 0). \end{cases}$$

Note that if $h = f$, then $J(h) = f$. If i is the least value such that $h(i) = 0$, then $J(h) = g_i$.

So h contains a zero if and only if $R_{\text{LLPO}}(J(h)) \neq R_{\text{LLPO}}(f)$. We can use R_{LLPO} and J to compute R_{LPO}.

The underlying computability theory

Why does $\mathrm{RCA}_0^\omega \vdash (\mathrm{LLPO}) \to (\mathrm{LPO})$ when $\mathrm{LPO} \not\leq_{\mathrm{W}} \mathrm{LLPO}$?

The two approaches yield different information.

$\mathrm{LPO} \not\leq_{\mathrm{W}} \mathrm{LLPO}$ because there are no fixed computable pre-processing and post-processing functionals that can (uniformly) convert every realizer for LLPO into a realizer for LPO.

$\mathrm{RCA}_0^\omega \vdash (\mathrm{LLPO}) \to (\mathrm{LPO})$ because given any realizer for LLPO we can compute a realizer for LPO. In the proof, when we said "suppose $R_{\mathrm{LLPO}}(f) = 0$" we made a non-uniform choice.

What about the other half of the equivalence?

Prop: $\mathsf{RCA}_0^\omega \vdash (\mathsf{LPO}) \to (\mathsf{LLPO})$.

The idea of the proof:

RCA_0^ω can prove the existence of the pre-processing and post-processing functionals in our second proof of $\mathsf{LLPO} \leqslant_W \mathsf{LPO}$. The implication follows by composition of functionals.

The post-processing functional in the first proof was not total, so that argument does not work in the higher order reverse mathematics setting.

Three questions, one answer.

- Are all Skolemized functional existence statements (SFEs) corresponding to WKL_0 statements equivalent to (LPO) in higher order reverse mathematics?

  No. Kohlenbach [5] noted statements about moduli of uniform continuity that are conservative over WKL_0.

- Which SFEs corresponding to WKL_0 are equivalent to (LPO)? Are the others weaker and equivalent to each other, or is there a low-level higher-order zoo?

- What about SFEs corresponding to $\Pi^1_1-\mathsf{CA}_0$ (and the Suslin functional) as compared to those corresponding to theorems equivalent to ATR_0?

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