My favorite theorem

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In this talk...

I want to tell you about my favorite theorem. I don't intend to tell you about my research at all. Congrats – you dodged a bullet there.

Theorem (Marker, '83)

If X computes an enumeration of the Scott set $\mathcal{S}$, then X also computes an effective enumeration of the Scott set $\mathcal{S}$.

I'll soon tell you what all the words here mean, but even without that, this should be remarkable. Effectivity of *anything* should never be "free".

Definition

A Scott set $\mathcal{S}$ is a nonempty subset of $P(\omega)$ with the following properties:

- If $X \in \mathcal{S}$ and $Y \leq_T X$ then $Y \in \mathcal{S}$
- If $X, Y, \in \mathcal{S}$ then $X \oplus Y \in \mathcal{S}$.
- If $T \subseteq 2^{<\omega}$ is infinite and $T \leq_T X \in \mathcal{S}$, then there is some $Y \in \mathcal{S}$ which is a path through T.

Definition

$E \subseteq \omega^2$ is an enumeration of the Scott set $\mathcal{S}$ if $\{E_i \mid i \in \omega\} = \mathcal{S}$ where $E_i = \{j \mid (i, j) \in E\}$.

Observation

For any X in a Scott set $\mathcal{S}$, there is a $Y >_T X$ in $\mathcal{S}$.

Effectively enumerating Scott sets

Definition

E is an effective enumeration of the Scott set $\mathcal{S}$ if E is an enumeration of $\mathcal{S}$ and there are computable functions witnessing the closure properties of the Scott set. i.e. There is a computable function $f(i, j)$ so that if T is an infinite tree in $2^{<\omega}$ and $T = \varphi_i(E_j)$, then $E_{f(i,j)}$ is a path through T.

So, enumerating a Scott set is just listing off its sets. Effectively enumerating it is to list off its sets and be able to computably *find* the columns that you know must be there somewhere.

Theorem (Marker, '83)

If X computes an enumeration of the Scott set $\mathcal{S}$, then X also computes an effective enumeration of the Scott set $\mathcal{S}$.

Effectivity for free! This is a statement that no computability theorist believes at first sight. After all, how could I possibly know which of the infinitely many columns is a path through T?

The trajectory of our journey

There are several components of this talk.

- The theory PA & overspill.
- Coding subsets of ω by elements in models of PA.
- The standard system of a model of PA, computable from a computation of the model.
- Matiyasevich's theorem on Hilbert's 10th problem.
- The standard system of a model of PA revisited.
- $\mathcal{S}$-Saturation and Homogeneity.
- Goncharov-Peretyat'kin on deciding homogeneous models.
- What does any of this have to do with Marker's theorem?
- Is there a direct proof? Inviting wild speculation into the computability-theoretic role of homogeneity/overspill/Hilbert's 10th.

The theory of PA and overspill

The theory PA is in the language $\{+, \cdot, 0, 1, <\}$ and it says the basic arithmetic facts (like distributivity, associativity, etc.) and the induction axioms:

Induction: For any formula $\varphi(x, \bar{y})$ and any tuple $\bar{a}$, either $\varphi(\mathcal{M}, \bar{a})$ is empty or it contains a least element.

Every model of PA starts with $0, 1, 2, \ldots$, so has an initial segment that looks like $\mathbb{N}$. Further, since the complement of $\mathbb{N}$ cannot have a least element (you can always subtract 1 and stay outside of $\mathbb{N}$), $\mathcal{M} \setminus \mathbb{N}$ is either empty or not definable.

Observation (Overspill Principle)

If $\mathcal{M} \models PA$ is not $\mathbb{N}$ and X is a definable subset of $\mathcal{M}$ which contains all of $\mathbb{N}$, then it must contain some element outside of $\mathbb{N}$.

Proof.

Otherwise X is a definition of $\mathbb{N}$ which violates an axiom of induction. □

Coding subsets of ω in models of PA

Definition

If $a \in \mathcal{M} \models \mathrm{PA}$, then $r(a) = \{n \mid \text{the } n^{\text{th}} \text{ prime divides } a\}$.
For $\mathcal{M} \models \mathrm{PA}$, we let $\mathcal{SS}(\mathcal{M}) = \{r(a) \mid a \in \mathcal{M}\}$.

Theorem (Scott-Tennenbaum)

If $\mathcal{M}$ is a nonstandard model of PA, then $\mathcal{SS}(\mathcal{M})$ is a Scott set.

Proof.

Let T be an infinite tree coded by $r(a)$. That is, $\sigma \in T$ if and only if $p_\sigma | a$ (the σth prime – using a bijection between $2^{<\omega}$ and $\mathbb{N}$). Then consider the set of n so that $\{\sigma : p_\sigma | a\}$ defines a tree up to length n and there is a string σ_n of length n on the tree coded by some number b. This is a definable set and includes every standard integer. By overspill, there is a nonstandard integer in this set. So, we get some b coding a path through the tree. □

$\mathcal{M}$ computes an enumeration of $\mathcal{SS}(\mathcal{M})$

If we have a computation of $\mathcal{M} \models PA$, then we can computably find, for each n, the n^{th} prime p_n. Further, for any $x \in \mathcal{M}$, we can check if p_n divides x (because it either divides x or one of it's p_n successors). So, for any $x \in \mathcal{M}$, we can uniformly compute the set $r(x)$.

Lemma

If X computes a copy of $\mathcal{M}$, then X computes an enumeration of $\mathcal{SS}(\mathcal{M})$. We call this the Standard Enumeration of $\mathcal{SS}(\mathcal{M})$ from the copy of $\mathcal{M}$.

Theorem (Tennenbaum)

There is no computable nonstandard model of PA.

Proof.

From a copy of the nonstandard model $\mathcal{M}$, we compute an enumeration of $\mathcal{SS}(M)$. But this is a Scott set, so it contains a non-computable element. □

Hilbert's 10th problem and Matiyasevich's theorem

Hilbert's 10th problem asked for an algorithm to determine whether or not a polynomial $p(\bar{z})$ has a solution in $\mathbb{N}$ (or $\mathbb{Z}$).

Definition

A set $X \subseteq \mathbb{N}^k$ is diophantine if there is some polynomial $p(\bar{x}, \bar{y})$ so that $X = \{\bar{x} \mid \exists \bar{y} p(\bar{x}, \bar{y}) = 0\}$

Theorem (Matiyasevich)

Every computably enumerable set is diophantine.

Theorem (Matiyasevich's theorem in PA)

If $\varphi(\bar{x})$ is a Σ_1^0-definable subset of $\mathcal{M} \models PA$, then there is some p so that $\{\bar{x} \mid \mathcal{M} \models \exists \bar{y} p(\bar{x}, \bar{y}) = 0\} = \varphi(\mathcal{M})$

A forumla is Σ_1^0 if it is formed from quantifier-free formulas and the quantifiers $\exists y$, $\exists y < a$ and $\forall y < a$.

Σ^0_1-definable sets are computably enumerable in M

Corollary

Let $\mathcal{M} \models PA$. Let A be a Σ^0_1-definable set. Then A is computably enumerable in any copy of $\mathcal{M}$.

Proof.

The problem in this lemma is that $\forall x < v$ is not obviously computable to check if v is a non-standard integer. In $\mathbb{N}$, bounded quantification is obviously computable: Just check the finitely many cases needed.
Solution: Replace the Σ^0_1 formula with an (equivalent) diophantine definition. There are no $\forall x < v$'s in the diophantine definition.

□

Corollary

If X is Δ^0_1-definable in a model of PA, then it is computable in any copy of $\mathcal{M}$.

First huge shock: We get effectivity free here.

Theorem (Solovay)

If $\mathcal{M}$ is a nonstandard model of PA, then the Standard Enumeration of $\mathcal{SS}(\mathcal{M})$ is M-effective. i.e., $\mathcal{M}$ can compute the functions witnessing the closedness of the Scott set. So, $\mathcal{M}$ computes an effective enumeration of $\mathcal{SS}(\mathcal{M})$.

Proof of Solovay's theorem.

Given $a \in \mathcal{M}$ (where we think of a as coding the the tree T), there is some $\sigma \in 2^{<M}$ and $v \in \mathcal{M}$ so that

$$(\forall n < |\sigma|)\, p_{\sigma\restriction n}|a \wedge$$

$$(\forall \tau \in 2^{|\sigma|+1})\neg(\forall n < |\tau|)\, p_{\tau\restriction n}|a \wedge$$

$$(\forall i < |\sigma|)(p_i|v \leftrightarrow \sigma(i) = 1)$$

In words: σ is a longest node on the tree and v codes σ.

□

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$\mathcal{S}$-saturation

Definition

Let $\mathcal{S} \subseteq P(\omega)$. $\mathcal{M}$ is $\mathcal{S}$-saturated if

- Every type $p(\bar{x})$ realized in $\mathcal{M}$ is computable in some $X \in \mathcal{S}$.
- If $p(\bar{x}, \bar{y})$ is a type computable in some $X \in \mathcal{S}$ and $\bar{a} \in \mathcal{M}$ is so that $p(\bar{x}, \bar{a})$ is consistent, then p is realized in $\mathcal{M}$.

Usually model-theorists consider full saturation, i.e., $\mathcal{S} = P(\omega)$.

Lemma

If $\mathcal{S}$ is a countable Scott set and T is a complete theory in $\mathcal{S}$, then T has a unique countable $\mathcal{S}$-saturated model.

Goncharov-Peretyat'kin on deciding homogeneous structures

Theorem (Goncharov and Peretyat'kin, 78)

Let A be ω-homogeneous. Let E be a d-enumeration of the types realized in A. Suppose further that E has the d-effective extension property. Then there is a copy of A which is d-decidable.

Definition

An enumeration E of types has the d-effective extension property if there is a d-computable function $g(i, j)$ so that if $p(\bar{x}) = E_i$ and $\varphi_j(\bar{x}, y)$ is consistent with p, then $E_{g(i,j)}$ is a type containing p and φ_j.

Deciding $\mathcal{S}$-saturated models

Theorem (Marker)

If $T \in \mathcal{S}$ and E is an enumeration of $\mathcal{S}$, then there is a copy of the countable $\mathcal{S}$-saturated model of T whose elementary diagram is computable in E (i.e. $\mathcal{M}$ is E-decidable).

From a type $p(\bar{x})$ and a formula $\varphi(\bar{x}, y)$ it is computable to produce a type containing both. The problem is in finding an index for it in the enumeration. But we can do this easily if we produce a new enumeration of the T-types in $\mathcal{S}$ wherein we explicitly build many sets just to be these "extension types". Then, since $\mathcal{S}$ is closed under Turing reduction, this is also an enumeration of the T-types in $\mathcal{S}$.

So, starting with an enumeration of $\mathcal{S}$, we computably produce an enumeration of the T-types in $\mathcal{S}$ and then a second enumeration of the T-types in $\mathcal{S}$ along with a function witnessing the effective extension property. Apply Goncharov-Peretyat'kin.

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My favorite theorem - reminder slide

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Marker's proof

Proof.

Let E be the X-computable enumeration of $\mathcal{S}$. There is some completion T of PA contained in $\mathcal{S}$. Then X also computes (by the last theorem) the elementary diagram of an $\mathcal{S}$-saturated model $\mathcal{M}$ of T.

Now, let $R = \mathcal{SS}(\mathcal{M})$. This R is an X-effective enumeration of a Scott set. We now only need:

Lemma

If M is an $\mathcal{S}$-saturated model of PA, then $\mathcal{S} = \mathcal{SS}(\mathcal{M})$.

Proof.

$\mathcal{SS} \subseteq \mathcal{S}$, since $r(a)$ is computable in $\mathrm{tp}(a)$. $\mathcal{S} \subseteq \mathcal{SS}$, since for any set $A \in \mathcal{S}$, we can cook up a type in $\mathcal{S}$ containing all the formulae $p_i | x$ if and only if $i \in A$. $\square$

$\square$

Some general comments

Marker's theorem does not mention PA or model theory, yet the only known proof involves specifically looking at models of PA and looking at $\mathcal{S}$-saturated models.

It would be fascinating to see if there were a purely computability-theoretic proof. If so, what serves the role of homogeneity/saturation? What fills the role of Matiyasevich's theorem?!

It follows from results of Lachlan and Soare that there is an enumeration X of a jump ideal $\mathcal{S}$ so that X does not compute any enumeration of $\mathcal{S}$ which can find jumps effectively. So, this property of "free" uniformity is unique for finding paths through trees.

The end

Thank you!