Galvin's problem in higher dimensions

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Outline

Ramsey's theorem

Theorem (Ramsey)

Suppose $c : [\mathbb{N}]^2 \to 2$ *is any function. Then there is an infinite* $X \subseteq \mathbb{N}$ *such that* c *is constant on* $[X]^2$.

Theorem (Ramsey)

Suppose $k, l \geq 1$ *are natural numbers. If* $c : [\mathbb{N}]^k \to l$ *is any function, then there is an infinite set* $X \subseteq \mathbb{N}$ *such that* c *is constant on* $[X]^k$.

Expansion Problems

- A consequence of Ramsey's theorem may be a little less well known: all finitary relations on $\mathbb{N}$ can be classified modulo restriction to an infinite subset of $\mathbb{N}$.
- Recall there are "too many" binary relations on $\mathbb{N}$ to classify up to isomorphism.
- For example there are continuum many pairwise non-isomorphic linear orders on $\mathbb{N}$.

Theorem (Ramsey)

*Suppose $R \subseteq \mathbb{N}^2$ is any relation. Then there is an infinite set $M \subseteq \mathbb{N}$ such that $R \cap M^2$ is **equal** to one of the following relations restricted to M: $\top$, $\bot$, $=$, $\neq$, $<$, $>$, $\leq$, $\geq$.*

- There is an analogous result for subsets of $\mathbb{N}^k$ for any finite k.
- For all finite k, the relations are quantifier free definable using $=$ and $<$.

Definition

Let A and B be structures. For natural numbers $k, l, t \geq 1$, the notation

$$B \longrightarrow (A)^k_{l,t}$$

means that for every coloring $c : [B]^k \to l$, there exists a substructure C of B such that C is isomorphic to A and $\left|c''[C]^k\right| \leq t$.

- Suppose that C is some class of structures and that A is a structure that embeds into every member of C.

Ramsey degree

For a natural number $k \geq 1$, the k-dimensional Ramsey degree of A within C is the the smallest natural number $t_k \geq 1$ (if it exists) such that $B \to (A)^k_{l,t_k}$, for every natural number $l \geq 1$ and for every structure $B \in C$. When no such t_k exists, we say that the k-dimensional Ramsey degree of A within C is infinite or does not exist.

Theorem (Ramsey)

For each $k \geq 1$, the k-dimensional Ramsey degree of $\mathbb{N}$ within the class of all infinite sets is 1.

- Suppose that C is some class of structures and that A is a structure that embeds into every member of C.

Expansion Problem

Suppose that $R_1, \ldots, R_m$ are finitely many finitary relations on the structure A. The relations $R_1, \ldots, R_m$ are said to solve the expansion problem for A within the class C if for every structure $B \in C$ and every finitary relation S on B, there exists a substructure C of B and an isomorphism $\varphi : A \to C$ such that the restriction of S to C is quantifier free definable from the images of $R_1, \ldots, R_m$ under φ.

Theorem (Ramsey)

The relations $<$ and $=$ solve the expansion problem for $\mathbb{N}$ within the class of all infinite sets.

These are equivalent problems

- Solving the expansion problem for A within C for k-ary relations is equivalent to finding the k-dimensional Ramsey degree of A within C.
- This notion of k-dimensional Ramsey degree is distinct from the notions of Ramsey degree in the context of Fraïssé theory.
- Special cases of this problem appear in topological dynamics in the guise of computing the universal minimal flows of various automorphism groups.

Expansion problem for the rationals

- The 2-dimensional Ramsey degree of $\langle \mathbb{Q}, < \rangle$ within the class $\{\langle \mathbb{Q}, < \rangle\}$ is not 1.

Sierpinski's coloring

Let $<_{\mathtt{wo}}$ be a well-ordering of $\mathbb{Q}$. Define $s : [\mathbb{Q}]^2 \to \{0, 1\}$ by

$$s(\{p, q\}) = \begin{cases} 0 & \text{if } < \text{ and } <_{\mathtt{wo}} \text{ disagree on } \{p, q\} \\ 1 & \text{if } < \text{ and } <_{\mathtt{wo}} \text{ agree on } \{p, q\}, \end{cases}$$

for any $\{p, q\} \in [\mathbb{Q}]^2$.

- For any $X \subseteq \mathbb{Q}$:
  - if s is constantly 1 on $[X]^2$, then X is well-ordered by the usual ordering $<$;
  - if s is constantly 0 on $[X]^2$, then X is well-ordered by the reserve ordering $>$.
- Thus if $\langle X, < \rangle$ contains a $\mathbb{Z}$-chain, then s takes both colors on $[X]^2$.

Theorem (Galvin)

Suppose $l \in \mathbb{N}$*. If* $c : [\mathbb{Q}]^2 \to \{0, \ldots, l\}$ *is any function, then there exists* $X \subseteq \mathbb{Q}$ *such that* $\langle X, < \rangle$ *isomorphic to* $\langle \mathbb{Q}, < \rangle$ *and* c *takes at most 2 values on* $[X]^2$*.*

- In other words, the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, < \rangle$ within $\{\langle \mathbb{Q}, < \rangle\}$ is precisely 2.

Theorem (Laver; Devlin)

For every $k \geq 1$, the k-dimensional Ramsey degree of $\langle \mathbb{Q}, < \rangle$ within the class of all non-empty dense linear orders without endpoints exists.

This degree t_k is given by the following formula: $t_1 = 1$, and for $k > 1$, $t_k = \sum_{l=1}^{k-1} \binom{2k-2}{2l-1} \cdot t_l \cdot t_{k-l}$.

- The sequence $\{t_k\}_{k \geq 1}$ are called the odd tangent numbers because $t_k = T_{2k-1}$, where $\tan(z) = \sum_{n=0}^{\infty} \frac{T_n}{n!} z^n$.

Corollary

Let $<_{\mathtt{wo}}$ be any well-ordering of $\mathbb{Q}$. Then the relations $<, =$, and $<_{\mathtt{wo}}$ solve the expansion problem for the structure $\langle \mathbb{Q}, < \rangle$ within the class of all non-empty dense linear orders without endpoints.

The topological structure of the rationals

- Let $\mathcal{T}_{\mathbb{R}}$ denote the usual topology of the real numbers, and $\mathcal{T}_X$ its restriction to any $X \subseteq \mathbb{R}$.
- It is not true that if $X \subseteq \mathbb{Q}$ is order isomorphic to $\mathbb{Q}$, then X is homeomorphic to $\mathbb{Q}$.
- Easy exercise: construct $X \subseteq \mathbb{Q}$ which is order-isomorphic to $\mathbb{Q}$, but so that every point is isolated.

Theorem (Sierpiński)

$\langle X, \mathcal{T} \rangle$ is homeomorphic to $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ if and only if $\langle X, \mathcal{T} \rangle$ is non-empty, countable, metrizable, and dense-in-itself.

- It turns out that the expansion problem for $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within the class $\{\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle\}$ does not have any solution.

Theorem (Baumgartner [1])

Suppose $\langle X, \mathcal{T} \rangle$ is any Hausdorff space with $|X| = \aleph_0$. There is a coloring $c : [X]^2 \to \omega$ such that for any subspace $R \subseteq X$ that is homeomorphic to $\mathbb{Q}$, $c''[R]^2 = \omega$.

- For each natural number $l \geq 1$, define $d_l : [\mathbb{Q}]^2 \to l$ by $d_l(\{x, y\}) = c(\{x, y\}) \mod l$.
- If $X \subseteq \mathbb{Q}$ is homeomorphic to $\mathbb{Q}$, then d_l will take all l values on $[X]^2$.
- No finite list of finitary relations on $\mathbb{Q}$ will capture all binary relations on $\mathbb{Q}$ up to shrinking to a topological copy of $\mathbb{Q}$.

Theorem (Todorcevic and Weiss)

If $\langle X, d \rangle$ is a σ-discrete metric space, then there is a coloring $c : [X]^2 \to \omega$ such that $c''[Y]^2 = \omega$ for all $Y \subseteq X$ homeomorphic to $\mathbb{Q}$.

- How about the class of all uncountable sets of reals?

Galvin's Conjecture (1970s)

Suppose $X \subseteq \mathbb{R}$ is uncountable. For every natural number $l \geq 1$,

$$\langle X, \mathcal{T}_X \rangle \to \left(\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle \right)^2_{l,2}$$

Theorem (R. and Todorcevic [2])

If there is a Woodin cardinal, then the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within the class of all uncountable sets of reals is 2.

- Note this includes sets of reals of size $\aleph_1$. Recall $\aleph_1 \not\rightarrow [\aleph_1]^2_{\aleph_1}$.

- This result solves the expansion problem for **binary relations** for the structure $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within the class of all uncountable sets of real numbers.

Theorem (R.+Todorcevic [2])

Assume that there is a Woodin cardinal. Let $<_{\mathtt{wo}}$ be any well-ordering of $\mathbb{R}$. Then for every uncountable $X \subseteq \mathbb{R}$ and every binary relation $M \subseteq X^2$, there exists a set $Y \subseteq X$, which is homeomorphic to $\mathbb{Q}$, such that $M \cap Y^2$ is quantifier free definable from the restrictions of $<_{\mathtt{wo}}$, $<$, and $=$ to Y.

- We can go beyond just sets of reals.

Theorem (R.+Todorcevic [2])

If there is a proper class of Woodin cardinals, then the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within the class of all non-σ-discrete metric spaces is equal to 2.

Definition

Let $\langle X, \mathcal{T} \rangle$ be a topological space. A base $\mathcal{B} \subseteq \mathcal{T}$ is said to be point-countable if for each $x \in X$, $\{U \in \mathcal{B} : x \in U\}$ is countable.

Definition

A topological space $\langle X, \mathcal{T} \rangle$ is said to be left-separated if there exists a well-ordering $<_{\mathtt{wo}}$ of X so that for each $x \in X$, $\{y \in X : y <_{\mathtt{wo}} x\}$ is a closed set.

Theorem (R.+Todorcevic [2])

If there is a proper class of Woodin cardinals, then the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within the class of all regular, non-left-separated spaces with point-countable bases is at most 2.

Definition

δ is a Woodin cardinal if for every $f : \delta \to \delta$, there exists $\kappa < \delta$ such that κ is closed under f and there exists $j : \mathbf{V} \prec \mathbf{M}$ with $\mathrm{crit}(j) = \kappa$ and $V_{j(f)(\kappa)} \subseteq \mathbf{M}$.

- We need a δ such that the countable stationary tower up to δ is precipitous.

Definition

Let δ be a strongly inaccessible cardinal. As usual, V_δ denotes $\{a : \mathrm{rank}(a) < \delta\}$. The countable stationary tower up to δ, denoted $\mathbb{Q}_{<\delta}$, is defined to be the collection of all $\langle A, S \rangle \in V_\delta$ such that A is a non-empty set and $S \subseteq [A]^{<\aleph_1}$ is stationary in $[A]^{<\aleph_1}$.

An ordering on $\mathbb{Q}_{<\delta}$ is defined as follows. For $\langle A, S \rangle, \langle B, T \rangle \in \mathbb{Q}_{<\delta}$, define $\langle B, T \rangle \leq \langle A, S \rangle$ to mean that $B \supseteq A$ and $T \subseteq \{M \in [B]^{<\aleph_1} : M \cap A \in S\}$.

Definition

Define a two-player game $\mathfrak{D}(\delta)$ as follows. Two players Empty and Non-Empty take turns playing conditions in $\mathbb{Q}_{<\delta}$, with Empty making the first move. When one of the players has played $\langle A_n, S_n \rangle \in \mathbb{Q}_{<\delta}$, his opponent is required to play $\langle A_{n+1}, S_{n+1} \rangle \leq \langle A_n, S_n \rangle$. Thus each run of the game produces a sequence

Empty	$\langle A_0, S_0 \rangle$		$\langle A_2, S_2 \rangle$		$\cdots$
Non-Empty		$\langle A_1, S_1 \rangle$		$\cdots$	

such that for each $n \in \omega$, $\langle A_{2n}, S_{2n} \rangle$ has been played by Empty, $\langle A_{2n+1}, S_{2n+1} \rangle$ has been played by Non-Empty and $\langle A_{n+1}, S_{n+1} \rangle \leq \langle A_n, S_n \rangle$. Non-Empty wins this particular run of $\mathfrak{D}(\delta)$ if and only if there exists a sequence $\langle N_l : l \in \omega \rangle$ such that $\forall l \in \omega\, [N_l \in S_l]$ and $\forall k \leq l\, [N_k = N_l \cap A_k]$.

- We need a δ such that Empty does not have a winning strategy in $\Game(\delta)$.

- When C is the class of all regular, non-left-separated spaces with point-countable bases, then the large cardinal hypothesis can be weakened to the following: for every ordinal α, there exists an inner model N of ZFC such that $V_\alpha \subseteq N$ and there is a Woodin cardinal greater than α in N.
- This weakening is implied by each of the following: existence of one strongly compact cardinal, PFA, PID.
- The weakening does not even imply the existence of an inaccessible cardinal in $\mathbf{V}$.
- Woodin showed that this weakening is equivalent to the statement that $\mathbf{\Sigma}^1_2$-determinacy holds in $\mathbf{V}$ and all of its set generic extensions.

- When C is the class of all uncountable sets of reals, then the large cardinal hypothesis can be weakened to the following: there is an inner model containing all sets of reals with at least one Woodin cardinal in it.
- Actually, if one is only interested in consistency strength, then an upper bound in this case is one measurable cardinal.
- If there is a precipitous ideal on ω_1, then the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within the class of all uncountable sets of reals is 2.

Higher dimensions

- A generalization of Sierpinski's coloring shows that the number of unavoidable colors in dimension k (on a topological copy of $\mathbb{Q}$) is $k!(k-1)!$.
- Do large cardinals imply that this number can always be achieved for any coloring of $[\mathbb{R}]^3$?

Theorem (R.+Todorcevic [3])

Let $n \in \omega$. Let $\langle X, \mathcal{T} \rangle$ be any Hausdorff space with $|X| = \aleph_n$. There is a coloring $c : [X]^{n+2} \to \omega$ such that for any subspace $R \subseteq X$ that is homeomorphic to $\mathbb{Q}$, $c''[R]^{n+2} = \omega$.

- The case $n = 0$ is precisely Baumgartner's theorem.
- So the $n + 2$-dimensional expansion problem for the space $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within the class of sets of real numbers of size at most $\aleph_n$ does not have any solution.

Corollary

Let $n \in \omega$. Suppose C is any class of topological spaces. If C contains any Hausdorff space of cardinality at most $\aleph_n$, then the $n+2$-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ within C does not exist.

Corollary

If $\langle \mathbb{R}, \mathcal{T}_{\mathbb{R}} \rangle \to \left(\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle \right)^3_{l,12}$, for all $1 \leq l < \omega$, then CH fails. For any $k \geq 1$, if for every $1 \leq l < \omega$, $\langle \mathbb{R}, \mathcal{T}_{\mathbb{R}} \rangle \to \left(\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle \right)^k_{l,k!(k-1)!}$, then $|\mathbb{R}| \geq \aleph_{k-1}$. If the k-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ in $\{\langle \mathbb{R}, \mathcal{T}_{\mathbb{R}} \rangle\}$ exists for every natural number $k \geq 1$, then $2^{\aleph_0} \geq \aleph_{\omega+1}$.

- A key combinatorial aspect of the proof is a classical set mapping theorem of Kuratowski.

Lemma (Kuratowski)

For each $n \in \omega$, *there exists* $f_n : [\omega_n]^{n+1} \to [\omega_n]^{<\aleph_0}$ *such that:*

1. $\forall s \in [\omega_n]^{n+1} \, [f_n(s) \subseteq \max(s)]$;
2. $\forall t \in [\omega_n]^{n+2} \exists \alpha \in t \, [\alpha < \max(t)$ *and* $\alpha \in f_n(t \setminus \{\alpha\})]$.

Questions

Question

What is the largest class of topological spaces within which the k-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_{\mathbb{Q}} \rangle$ is equal to $k!(k-1)!$?

Bibliography

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