A tour of monadic dividing lines

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Overview

Definition and examples

Definition

A theory T is *monadically* $\mathcal{P}$ if any expansion of T by arbitrarily many unary predicates remains $\mathcal{P}$.

- We will consider the following hierarchy: monadically NIP, monadically stable, monadically NFCP, cellular.

Examples

- Cellular: An equivalence relation consisting of infinitely many finite classes.
- Monadically NFCP: Bounded-degree graphs.
- Monadically stable: Refining equivalence relations.
- Monadically NIP: $(\mathbb{Q}, <)$ and various tree-like structures.
- Not monadically NIP: Essentially anything with a non-unary function (e.g. vector spaces), cross-cutting equivalence relations, and the generic permutation.

More on monadic properties

- Nothing beyond monadic NIP.
- Give significant information beyond their non-monadic counterparts.
- Several applications in the combinatorics of hereditary classes and countable structures.
- Preserved by taking substructure, and so about universal theories.
- Actually agree with non-monadic counterparts in universal theories [BL22b].
- We will focus on three characterizations: independence relations, decompositions, and canonical obstructions.

Characterizations

- Applications: Sparse graph classes [AA14], twin-width for hereditary (ordered) graph classes [ST21]
- We characterize monadically NIP theories in the following ways [BL21], building on [BS85] and [She86].

1. The behavior of independence
2. A forbidden configuration
3. Decompositions of models
4. Type counting/width
5. The behavior of indiscernibles

- These are all characterizations of the theory T itself rather than unary expansions.
- An intuition is that models of monadically NIP theories are 1-dimensional, or alternatively are order-like (or tree-like).

Independence: The f.s. dichotomy

- Finite satisfiability gives a (possibly asymmetric) notion of independence in any theory.

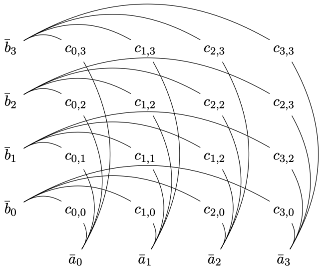
Definition ([She86])

Let $A \mathop{\smile\!\!\!\!\!\!|}^{fs}_M B$ mean that $tp(A/MB)$ is finitely satisfiable in M.
A theory T *has the f.s.-dichotomy* if given $A \mathop{\smile\!\!\!\!\!\!|}^{fs}_M B$, then for any c, $cA \mathop{\smile\!\!\!\!\!\!|}^{fs}_M B$ or $A \mathop{\smile\!\!\!\!\!\!|}^{fs}_M Bc$.

Forbidden configuration: An infinite grid

Lemma (mostly [She86])

If T does not have the f.s. dichotomy, then some model of T codes an infinite grid (on tuples).



- So monadically NIP $\Rightarrow$ f.s. dichotomy.
- If T is a universal theory then we can code grids by a boolean combination of existentials [BL22b].

Decomposability: M-f.s. sequences

Definition

Given a model M, $(A_i : i \in I)$ is an *M-f.s. sequence* if $A_i \mathop{\smile\!\!\!\!\!\!\;|}^{fs}_M \{A_{<i}\}$.

A *linear decomposition* of N is a partition $N = \sqcup_i A_i$ and a model M (not necessarily in N) such that $(A_i : i \in I)$ is an M-f.s. sequence.

Lemma ([She86])

If T has the f.s. dichotomy, then any partial linear decomposition of $N \models T$ can be extended to a full linear decomposition of N.

- The f.s. dichotomy is exactly what we need to inductively extend, one point at a time.

Monadic stability

Theorem ([BS85])

For a complete theory T, the following are equivalent.

1. *T is monadically stable*
2. *T is monadically NIP and stable*
3. *Models of T are tree-decomposable*
4. *T is stable and forking is transitive on singletons and totally trivial.*

- Totally trivial forking: If $A \not\mathop{\smile\hskip-0.9em\vert}_C B$, then there are $a \in A, b \in B$ such that $a \not\mathop{\smile\hskip-0.9em\vert}_C b$. (Vector spaces are a non-example.)

MORE MONADIC STABILITY

- Triviality on transitivity mean forking-dependence yields on equivalence relation on singletons.
- Structure (for uncountable models): Tree decompositions
- Non-structure: the order property (by an atomic formula [NMP$^+$21])
- Applications: ω-categorical classification [Lac92], sub-exponential growth rates of ω-categorical structures [Bra22]

Monadic NFCP

- Equivalence relation with arbitrarily large finite classes as a paradigm of stable but not NFCP.
- So an equivalence relation with infinitely many infinite classes is not monadically NFCP.

Theorem (mostly [Las13])

The following are equivalent for a complete theory T.

1. *T is monadically NFCP.*
2. *T mutually algebraic.*
3. *T is weakly minimal and forking is totally trivial.*
4. *Models of T can be partitioned into a sunflower with countable pieces (MA-connected components).*
5. *See next slide for non-structure characterizations.*

More on monadic NFCP

- Weak minimality: forking dependence just comes from algebraic closure.
- Mutual algebraicity: definable sets come from bounded-degree relations. In particular, after naming $|\mathcal{L}|$-many (or so) constants, models are quantifier-free interdefinable with bounded-degree structures.
- Structure: bounded-degree structures naturally split into countable connected components.
- For non-structure, assume a finite language. (T is monadically NFCP iff every finite reduct is.)
- If a universal T is monadically stable but not monadically NFCP, then some models defines an infinite equivalence relation (by a universal formula) [BL22a] (building on [LT20]).
- Robust closure under expansions

Cellularity

- Partitions like monadic NFCP, but with finiteness conditions.
- Does not encode a linear order or infinite partition.
- Preserved by finite monadic expansions.
- Equivalent to ω-categorical and an independence condition: ω-stable with Morley rank 1, and trivial algebraic closure [Sch90].
- If T is monadically NFCP but non-cellular, then every model admits an elementary extension containing infinitely many infinite new MA-connected components [BL22c].

Applications of cellularity

- Counting countable structures of a given age [MPW92]
- Counting substructures of a countable structure [LM96]
- Counting finite structures in a hereditary class [LT22].
- Counting structures bi-embeddable with a given countable structure [BL22a].

Conjecture (extending Pouzet-Sauer-Thomassé [LPSW21])

Given an age $\mathcal{A}$, let $|Mod(\mathcal{A})/{\equiv}|$ count the bi-embeddability classes of countable structures of age $\mathcal{A}$. Then $|Mod(\mathcal{A})/{\equiv}| \in \{1, \aleph_0, \aleph_1, 2^{\aleph_0}\}$. Furthermore, these cases correspond to cellular, (monadically) stable, (monadically) NIP, and everything else.

References I

- Hans Adler and Isolde Adler.
  Interpreting nowhere dense graph classes as a classical notion of model theory.
  European Journal of Combinatorics, 36:322–330, 2014.

- Samuel Braunfeld and Michael C Laskowski.
  Characterizations of monadic NIP.
  Transactions of the AMS, Series B, 8:948–970, 2021.

- SAMUEL BRAUNFELD and MICHAEL C. LASKOWSKI.
  Counting siblings in universal theories.
  The Journal of Symbolic Logic, 87(3):1130–1155, 2022.

- Samuel Braunfeld and Michael C Laskowski.
  Existential characterizations of monadic NIP.
  arXiv preprint arXiv:2209.05120, 2022.

References II

- Samuel Braunfeld and Michael C Laskowski.
  Mutual algebraicity and cellularity.
  Archive for Mathematical Logic, pages 1–17, 2022.

- Samuel Braunfeld.
  Monadic stability and growth rates of ω-categorical structures.
  Proceedings of the London Mathematical Society, 124(3):373–386, 2022.

- John T Baldwin and Saharon Shelah.
  Second-order quantifiers and the complexity of theories.
  Notre Dame Journal of Formal Logic, 26(3):229–303, 1985.

- Alistair H. Lachlan.
  $\aleph_0$-categorical tree-decomposable structures.
  The Journal of Symbolic Logic, 57(2):501–514, 1992.

References III

Michael C Laskowski.
Mutually algebraic structures and expansions by predicates.
The Journal of Symbolic Logic, 78(1):185–194, 2013.

Michael C Laskowski and Laura L Mayer.
Stable structures with few substructures.
The Journal of Symbolic Logic, 61(3):985–1005, 1996.

Claude Laflamme, Maurice Pouzet, Norbert Sauer, and Robert Woodrow.
Siblings of an $\aleph_0$-categorical relational structure.
Contributions to Discrete Mathematics, 18(2), 2021.

Michael C Laskowski and Caroline A Terry.
Uniformly bounded arrays and mutually algebraic structures.
Notre Dame journal of formal logic, 61(2):265–282, 2020.

References IV

- Michael C Laskowski and Caroline A Terry.
  Jumps in speeds of hereditary properties in finite relational languages.
  Journal of Combinatorial Theory, Series B, 154:93–135, 2022.

- HD Macpherson, Maurice Pouzet, and Robert E Woodrow.
  Countable structures of given age.
  The Journal of Symbolic Logic, 57(3):992–1010, 1992.

- Jaroslav Nešetřil, Patrice Ossona de Mendez, Michał Pilipczuk, Roman Rabinovich, and Sebastian Siebertz.
  Rankwidth meets stability.
  In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2014–2033. SIAM, 2021.

References V

James H Schmerl.
Coinductive $\aleph_0$-categorical theories.
The Journal of Symbolic Logic, 55(3):1130–1137, 1990.

Saharon Shelah.
Monadic logic: Hanf numbers.
In *Around classification theory of models*, pages 203–223.
Springer, 1986.

Pierre Simon and Szymon Toruńczyk.
Ordered graphs of bounded twin-width.
arXiv preprint arXiv:2102.06881, 2021.