

Some results on the structure of Weihrauch degrees

Manlio Valenti
manlio.valenti@wisc.edu

UW Logic Seminar, 2022

Computational problems

Given an instance of a problem, produce a solution.

Even $\forall\exists$ theorems can be seen as computational problems!

$$(\forall X)(\varphi(X) \rightarrow (\exists Y)\psi(X, Y))$$

We want to focus on problems on the Baire space $\mathbb{N}^{\mathbb{N}}$.

$$f \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$$

$$f : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \wp(\mathbb{N}^{\mathbb{N}})$$

$$f : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$$

Computability on $\mathbb{N}^{\mathbb{N}}$

Computability for functions $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ is induced by computability on \mathbb{N} using monotone partial computable functions (on \mathbb{N}).

Computability on multi-valued functions $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$:

$F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ is a realizer for $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ if

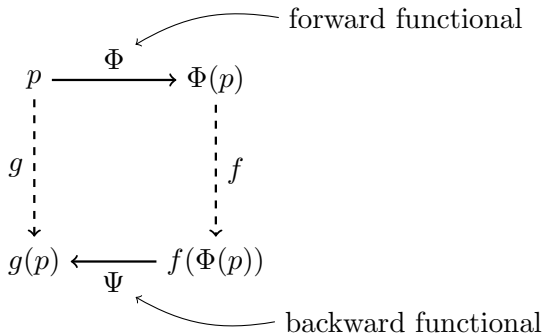
$$(\forall p \in \text{dom}(f))(F(p) \in f(p))$$

f is computable if it has a computable realizer, i.e. a computable choice function

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{F} & \mathbb{N}^{\mathbb{N}} \\ \text{id} \downarrow & & \downarrow \text{id} \\ \mathbb{N}^{\mathbb{N}} & \xrightarrow{f} & \mathbb{N}^{\mathbb{N}} \end{array}$$

Reducibilities on computational problems

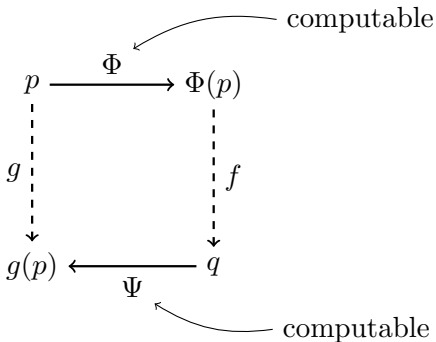
Diagram of a generic $g \leq f$:



According to the properties of Φ and Ψ we get different reductions.

Weihrauch reducibility

The forward and backward functional are partial computable functions on $\mathbb{N}^{\mathbb{N}}$.

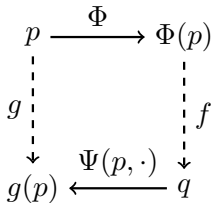


Weihrauch reducibility

The forward and backward functional are partial computable functions on $\mathbb{N}^{\mathbb{N}}$.

$g \leq_W f : \iff$ there are computable $\Phi, \Psi : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ s.t.

- Given $p \in \text{dom}(g)$, $\Phi(p) \in \text{dom}(f)$
- Given $q \in f(\Phi(p))$, $\Psi(p, q) \in g(p)$



Strong Weihrauch reducibility:

$g \leq_{sW} f : \iff g \leq_W f$ and Ψ does not depend on p .

Represented spaces

Weihrauch reducibility is usually defined in a more general context.

A *represented space* is a pair (X, δ_X) where X is just a set and $\delta_X : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$ is a surjection.

Computability for problems on represented spaces is induced via the notion of realizer

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{F} & \mathbb{N}^{\mathbb{N}} \\ \delta_X \downarrow & & \downarrow \delta_Y \\ X & \xrightarrow{f} & Y \end{array}$$

$$(\forall p \in \text{dom}(f \circ \delta_X)) (\delta_Y F(p) \in f(p))$$

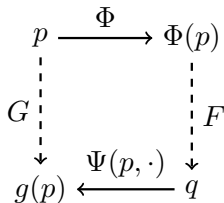
Represented spaces

Weihrauch reducibility is usually defined in a more general context.

A *represented space* is a pair (X, δ_X) where X is just a set and $\delta_X : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$ is a surjection.

Weihrauch reducibility $g \leq_W f$ is defined using names and realizers $F \vdash f, G \vdash g$:

- Given a name p for $x \in \text{dom}(g)$, $\Phi(p)$ is a name for some $z \in \text{dom}(f)$
- Given a name q for $w \in f(z)$, $\Psi(p, q)$ is a name for $y \in g(x)$



Weihrauch reducibility

Take-home messages:

- It is a uniform reduction with exactly one oracle call.
- It works on names
- If we are only interested on the structure of the degrees we do not need to talk about represented spaces.

While several authors have studied the degrees of specific problems, many questions on the structure of the degrees are open.

The structure of Weihrauch degrees

Is there a bottom element?

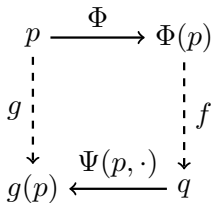
Yes, vacuously: \emptyset

Is there a top element?

No, under ZFC

A function with no realizer is a “natural” top element.

The existence of a realizer coincides with the existence of a choice function.



The structure of Weihrauch degrees

Is there a join?

Yes: for every f_0, f_1 , we define $f_0 \sqcup f_1 : \subseteq 2 \times \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ as

$$(f_0 \sqcup f_1)(i, p) := f_i(p)$$

with $\text{dom}(f_0 \sqcup f_1) := \text{dom}(f_0) \sqcup \text{dom}(f_1)$.

Why is this a join?

$$f_i \leq_W f_0 \sqcup f_1: p \mapsto (i, p)$$

If $f_i \leq_W h$ for $i < 2$ then $f_0 \sqcup f_1 \leq_W h$: assume $f_i \leq_W h$ via Φ_i, Ψ_i .

Given (i, p) we use Φ_i, Ψ_i to compute $f_i(p)$ using h .

The structure of Weihrauch degrees

Is there a meet?

Yes: for every f_0, f_1 , we define $f_0 \sqcap f_1 : \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ as

$$(f_0 \sqcap f_1)(p_0, p_1) := f_0(p_0) \sqcup f_1(p_1) = \{(i, r) : r \in f_i(p_i)\}$$

with $\text{dom}(f_0 \sqcap f_1) := \text{dom}(f_0) \times \text{dom}(f_1)$.

Proposition (Pauly; Brattka, Gherardi)

Meet and join distribute, hence the Weihrauch degrees form a distributive lattice.

The structure of Weihrauch degrees

How about infinite join/meet?

Theorem (Higuchi, Pauly)

No non-trivial countable suprema exists, i.e.

$$d = \sup\{d_n\}_{n \in \mathbb{N}} \iff (\exists N)(d = \sup\{d_n\}_{n < N})$$

Moreover, there is an infinite descending sequence $(b_n)_{n \in \mathbb{N}}$ in \mathcal{W} with no infimum.

Corollary (Higuchi, Pauly)

The Weihrauch degrees are not a \aleph_0 -complete join/meet semi-lattice.

Warning: the operations $\bigsqcup_{n \in \mathbb{N}} d_n$ and $\bigsqcap_{n \in \mathbb{N}} d_n$ are not degree-theoretic!

The structure of Weihrauch degrees

Is there a jump?

Well...there is a thing called “jump”

(Brattka, Gherardi, Marcone) For a problem f , f' works as follows:

input : a sequence $(p_n)_{n \in \mathbb{N}}$ converging to $p \in \text{dom}(f)$;

output : $f(p)$

However:

- it is not degree theoretic, let alone monotone
(it is monotone w.r.t. strong Weihrauch reducibility)
- it does not jump! (it is possible that $f' \equiv_W f$)

...What?

Intuitively, f' corresponds to:

given $(p_n)_{n \in \mathbb{N}}$ compute its limit p , and then solve $f(p)$. In symbols

$$f * \text{lim}$$

Historically, the name comes from:

- a basic analogy with the Turing jump for “simple” problems
- $\text{lim} \equiv_{sW} J$, where $J := p \mapsto p'$ is the Turing jump.

Open question: is there a *true* jump in the Weihrauch lattice/degrees?

Trying to build one “by hands”

It is not hard to define an operator that “jumps”. Example

$$f^1(\langle x, p \rangle) := \begin{cases} 1 \frown \Phi_x^f(p) & \text{if } \Phi_x^f(p) \downarrow \\ 0^{\mathbb{N}} & \text{otherwise} \end{cases}$$

where $\Phi_x^f(p)$ intuitively is a continuous functional that uses f as oracle.

Problem: this is not degree theoretic.

Proposition (Marcone, V.)

If an operator Λ is such that $\Lambda(f)$ computes the characteristic function of $\text{dom}(f)$ then Λ is not degree-theoretic.

Minimal degrees

Is there a minimal element above \emptyset ?

No

Proof (Dzhafarov, Lerman, Patey, Solomon).

Assume $\emptyset <_W f$. In particular, there is $p \in \text{dom}(f) \neq \emptyset$.

Define g as $\text{dom}(g) := \{p'\}$ and $g(p') := f(p)$.

$g \leq_W f$ as $p \leq_T p'$.

$f \not\leq_W g$ as $p' \not\leq_T p$, hence there is no computable Φ that map $\text{dom}(f)$ to $\text{dom}(g)$. □

We are heavily exploiting the complexity of the domain!

Minimal degrees

Is there a minimal element above id ?

id is the \leq_W -least problem with a computable input.

By adapting the classical proof of the existence of minimal pairs in the Turing degrees:

Proposition (Marcone, V.)

For every non uniformly computable $F: \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ there is a non uniformly computable function $G: \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ s.t. F and G form a minimal pair on the total deterministic degrees, i.e. for every $H: \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$, if $H \leq_W F$ and $H \leq_W G$ then $H \leq_W \text{id}$.

Can we do better?

Medvedev reducibility

Reducibility on subsets of $\mathbb{N}^{\mathbb{N}}$ (“mass problems”)

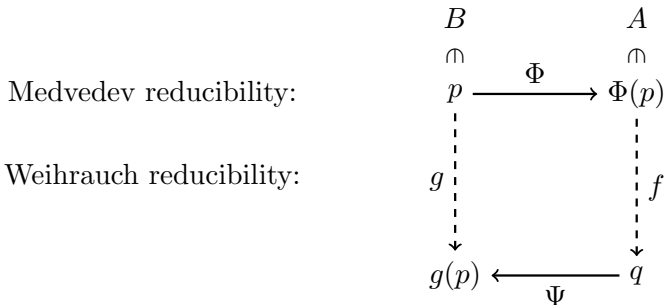
$$\begin{aligned} A \leq_M B &: \iff (\exists \Phi : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}} \text{ computable})(\Phi(B) \subset A) \\ &\iff (\exists \Phi : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}} \text{ computable})(\forall b \in B)(\Phi(b) \in A) \end{aligned}$$

The lower A is in the Medvedev degrees, the easier it is to uniformly compute an element of A .

Muchnik reducibility: non-uniform version of Medvedev

$$(\forall b \in B)(\exists \Phi : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}} \text{ computable})(\Phi(b) \in A)$$

Medvedev vs Weihrauch



“The first half of a Weihrauch reduction is a Medvedev reduction”.

$$g \leq_W f \Rightarrow \text{dom}(f) \leq_M \text{dom}(g)$$

This suggests a way to embed the Medvedev degrees in the Weihrauch degrees.

Medvedev \hookrightarrow Weihrauch

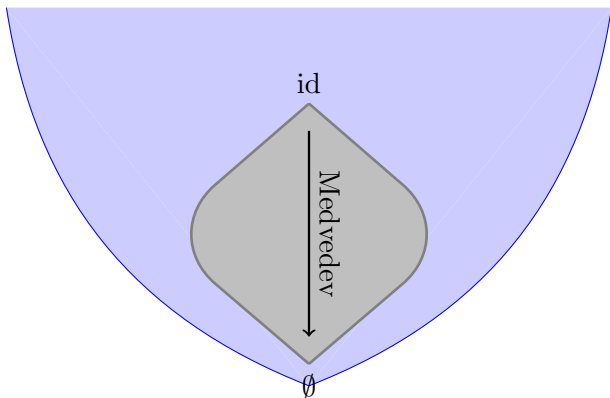
We map $A \subset \mathbb{N}^{\mathbb{N}}$ to $d_A: A \rightarrow \{0^{\mathbb{N}}\}$ defined as $d_A(p) := 0^{\mathbb{N}}$.

$B \leq_M A$ iff $d_A \leq_W d_B$:

$$\begin{array}{ccc} A & & B \\ \cap & & \cap \\ p & \xrightarrow{\Phi} & \Phi(p) \\ \vdots & & \vdots \\ d_A \downarrow & & \downarrow d_B \\ 0^{\mathbb{N}} & \xleftarrow{\Psi} & 0^{\mathbb{N}} \end{array}$$

This embedding reverses the Medvedev order!

Weihrauch degrees



Some results on the structure of Weihrauch degrees are obtained as corollaries of structural results on the Medvedev lattice.

Chains in the Weihrauch degrees

Let \mathcal{M} and \mathcal{W} be the Medvedev and Weihrauch degrees resp.

Corollary (of Terwijn)

Under $\text{ZFC} + 2^{<c} = c$, there is a chain of size 2^c in \mathcal{W} .

Let us write \mathcal{M}_0 for \mathcal{M} without the top element.

Proposition (Marcone, V.)

There is a chain in \mathcal{M}_0 of order type ω_1 with no upper bound.

The proof can be adapted to show that

Corollary (Marcone, V.)

There is a chain in \mathcal{W} of order type ω_1 with no upper bound.

Cofinality

A chain \mathcal{C} is cofinal in a poset \mathcal{P} if every element of \mathcal{P} is below some element of \mathcal{C} .

Proposition (Exercise)

Let \mathcal{T} denote the Turing degrees. The following are equivalent:

- CH;
- *There is a cofinal chain in \mathcal{T} (of order type ω_1).*

The same result holds for the Medvedev degrees.

Cofinality

Theorem (Marcone, V.)

The following are equivalent

1. CH;
2. *There is a cofinal chain in \mathcal{M}_0 (of order type ω_1).*

Cofinality

Theorem (Marcone, V.)

The following are equivalent

1. \mathfrak{CH} ; *There is a cofinal chain in \mathcal{T} of order type ω_1 ;*
2. *There is a cofinal chain in \mathcal{M}_0 (of order type ω_1).*

Proof: (Sketch).

(1) \Rightarrow (2): If $(d_\alpha)_{\alpha < \omega_1}$ is cofinal in \mathcal{T} then $(\{d_\alpha\})_{\alpha < \omega_1}$ is cofinal in \mathcal{M}_0 .

(2) \Rightarrow (1): Let $(A_\alpha)_{\alpha < \kappa}$ be cofinal in \mathcal{M}_0 . We can assume $\kappa = \omega_1$ as every mass problem has countably many singletons in its \leq_M -lower cone. Choosing $p_\alpha \in A_\alpha$, the set $\{p_\alpha : \alpha < \omega_1\}$ is cofinal in \mathcal{T} . We use $\{p_\alpha : \alpha < \omega_1\}$ to define a cofinal sequence $(q_\alpha)_{\alpha < \omega_1}$ in \mathcal{T} .

Cofinality

Proof: (cont).

For every $\alpha < \omega_1$ fix a *fundamental sequence* for α , i.e. a sequence $(\alpha[n])_{n \in \mathbb{N}}$ s.t.

- $(\forall n \in \mathbb{N})(\alpha[n] \leq \alpha[n+1] < \alpha)$
- $\alpha = \sup\{\alpha[n] + 1 : n \in \mathbb{N}\}$.

Define

- $q_0 := p_0$
- for $\alpha > 0$, $q_\alpha := \left(\bigoplus_{n \in \mathbb{N}} q_{\alpha[n]}\right) \oplus p_\alpha$.

Since $(q_\alpha)_{\alpha < \omega_1}$ is a chain and $p_\alpha \leq_T q_\alpha$, the claim follows. \square

Cofinality

The *set-cofinality* of a poset \mathcal{P} is the size of the smallest cofinal subset (every element of \mathcal{P} is below some element of the subset).

Theorem (Marcone, V.)

The set-cofinality of \mathcal{M}_0 is \mathfrak{c} .

The situation is different for the Weihrauch degrees:

Theorem (Marcone, V.)

There are no cofinal chains in \mathcal{W} and the set-cofinality is $> \mathfrak{c}$.

In particular, the last part follows from

Theorem (Marcone, V.)

For every family $\{f_p\}_{p \in \mathbb{N}^{\mathbb{N}}}$ of multi-valued functions, there is g s.t. for every p , $g \not\leq_{\mathcal{W}} f_p$.

Antichains

What do we know about antichains in \mathcal{M} or \mathcal{W} ?

Proposition (Marcone, V. generalizing Sorbi, Platek)

There are maximal antichains in \mathcal{M} of size κ , for every $1 \leq \kappa \leq \mathfrak{c}$ or $\kappa = 2^{\mathfrak{c}}$.

Open question: how about $\mathfrak{c} < \kappa < 2^{\mathfrak{c}}$?

This does not generalize to Weihrauch degrees!

Proposition (Dzhafarov, Lerman, Patey, Solomon)

For every countable family $\{f_n\}_{n \in \mathbb{N}}$ of non-trivial problems there is g s.t. for every n , $g \upharpoonright_{\mathcal{W}} f_n$. In particular, every countable antichain is extendible.

Maximal antichains

The result by (DLPS) cannot be extended to \mathfrak{c} -sized families:

Proposition (Marcone, V.)

There is a family $\{f_p\}_{p \in \mathbb{N}^{\mathbb{N}}}$ s.t. for every non-trivial g there is $p \in \mathbb{N}^{\mathbb{N}}$ s.t. $f_p \leq_W g$.

Proof.

For every $p \in \mathbb{N}^{\mathbb{N}}$, we define f_p with $\text{dom}(f_p) := \{p\}$ and $f_p(p) := \mathbb{N}^{\mathbb{N}}$. It is trivial to see that, for every non-empty g and for every $p \in \text{dom}(g)$, $f_p \leq_W g$. \square

Unfortunately, the above family cannot be refined to a maximal continuum-sized antichain!

Maximal antichains

Theorem (Marcone, V.)

If $\{f_p\}_{p \in \mathbb{N}^{\mathbb{N}}}$ is an antichain in \mathcal{W} s.t. $\{\text{dom}(f_p)\}_{p \in \mathbb{N}^{\mathbb{N}}}$ is not cofinal in \mathcal{M}_0 , then there is g s.t. for every $p \in \mathbb{N}^{\mathbb{N}}$, $g \upharpoonright_{\mathbb{W}} f_p$.

Proof: (Sketch).

Fix $A \subset \mathbb{N}^{\mathbb{N}}$ s.t. for every $p \in \mathbb{N}^{\mathbb{N}}$, $A \not\leq_{\mathcal{M}} \text{dom}(f_p)$. W.l.o.g. we can assume $|A| = \mathfrak{c}$.

We define a function g with $\text{dom}(g) := A$. This already guarantees that $f_p \not\leq_{\mathcal{W}} g$.

For the other direction, we define g so that $g((e, i) \hat{\cap} p)$ witnesses that $g \not\leq_{\mathcal{W}} f_p$ via Φ_e, Φ_i . We skip the details. \square

Maximal antichains

Since the set-cofinality of \mathcal{M}_0 is \mathfrak{c} :

Corollary (Marcone, V.)

No antichain $\{f_\alpha\}_{\alpha < \kappa}$ in \mathcal{W} with $\kappa < \mathfrak{c}$ is maximal.





Since no antichain in \mathcal{M}_0 is cofinal:

Corollary (Marcone, V.)

If $\{f_\alpha\}_{\alpha < \kappa}$ is an antichain in \mathcal{W} s.t. $\{\text{dom}(f_\alpha)\}_{\alpha < \kappa}$ is an antichain in \mathcal{M}_0 , then $\{f_\alpha\}_{\alpha < \kappa}$ is not maximal.

Open question: what happens if $\{f_p\}_{p \in \mathbb{N}^{\mathbb{N}}}$ is an antichain in \mathcal{W} and $\text{dom}(f_\alpha)$ is cofinal in \mathcal{M}_0 ?

References

-  Brattka, Vasco, Gherardi, Guido, and Pauly, Arno, *Weihrauch Complexity in Computable Analysis*, pp. 367–417, Springer International Publishing, Jul 2021, doi:10.1007/978-3-030-59234-9_11.
-  Higuchi, Kojiro and Pauly, Arno, *The degree structure of Weihrauch reducibility*, Logical Methods in Computer Science **9** (2013), no. 2:02, 1–17, doi:10.2168/LMCS-9(2:02)2013.
-  Sorbi, Andrea, *The Medvedev Lattice of Degrees of Difficulty*, Computability, Enumerability, Unsolvability (Cooper, S. B., Slaman, T. A., and Wainer, S. S., eds.), Cambridge University Press, New York, NY, USA, 1996, pp. 289–312.
-  Terwijn, Sebastiaan A., *On the Structure of the Medvedev Lattice*, The Journal of Symbolic Logic **73** (2008), no. 2, 543–558.