Toward deciding the $\forall\exists$-theory of the Σ^0_2-enumeration degrees

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(joint work with Goh, Ng and M. Soskova)

Most "natural" degree structures $\mathcal{D}$ are very complicated partial orders and usually follow this pattern:

- The first-order theory of the partial order $\mathcal{D}$ is undecidable.
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- The $\exists\forall\exists$-theory of $\mathcal{D}$ can "usually" be shown to be undecidable.

Definition

$A \leq_e B$ if there is an enumeration operator Φ with $A = \Phi(B)$, i.e., there is a c.e. set Φ of pairs (x, F) (of numbers x and finite sets F) denoting that for all x, $x \in A$ iff there is $(x, F) \in \Phi$ with $F \subseteq B$.

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In particular, we will focus on the degree structure $\mathcal{S}_e$ of the enumeration degrees of the Σ^0_2-sets, which coincides with the enumeration degrees $\boldsymbol{a} \leq \mathbf{0}'_e$. They form a densely ordered countable upper semilattice with least element $\mathbf{0}_e$ (the degree of the c.e. sets) and greatest element $\mathbf{0}'_e$ (the degree of $\overline{K}$).

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For $\mathcal{S}_e$, the $\exists$-theory is decidable by Lagemann (1972), whereas the $\exists\forall\exists$-theory is undecidable by Kent (2006).
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The full first-order theory is as complicated as first-order arithmetic by Ganchev/M. Soskova (2012).

However, the decidability of the ∀∃-theory of $\mathcal{S}_e$ remains open.

Deciding the ∀∃-theory of a degree structure $\mathcal{D}$ amounts to giving a uniform decision procedure to the following

Algebraic Problem (for deciding the ∀∃-theory of $\mathcal{D}$)

Given finite partial orders $\mathcal{P}$ and $\mathcal{Q}_i \supseteq \mathcal{P}$ (for $i \leq n$), does every embedding of $\mathcal{P}$ into $\mathcal{D}$ extend to an embedding of $\mathcal{Q}_i$ into $\mathcal{D}$ for some $i \leq n$ (where i may depend on the embedding of $\mathcal{P}$)?

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Two major subproblems of the ∀∃-theory of $\mathcal{S}_e$ are decidable:

Extension of Embeddings Problem

Given finite partial orders $\mathcal{P}$ and $\mathcal{Q} \supseteq \mathcal{P}$, does every embedding of $\mathcal{P}$ into $\mathcal{S}_e$ extend to an embedding of $\mathcal{Q}$ into $\mathcal{S}_e$?
(Lempp/Slaman/Sorbi 2005: complicated decision procedure)

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Lattice Embeddings Problem

Which finite lattices can be embedded into $\mathcal{S}_e$ (preserving join and meet)? (Lempp/Sorbi 2002: all finite lattices embed)

The main technical obstacles to deciding the $\forall\exists$-theory of $\mathcal{S}_e$ showed up first in the following

Theorem (Ahmad 1989 (cf. Ahmad/Lachlan 1998))

1. There is an *Ahmad pair* of Σ^0_2-enumeration degrees $(\boldsymbol{a}, \boldsymbol{b})$, i.e., there are incomparable degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ such that any degree $\boldsymbol{v} < \boldsymbol{a}$ is $\leq \boldsymbol{b}$.

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2. There is no *symmetric Ahmad pair* of Σ^0_2-enumeration degrees, i.e., there are no incomparable degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ such that any degree $\boldsymbol{v} < \boldsymbol{a}$ is $\leq \boldsymbol{b}$, and any degree $\boldsymbol{w} < \boldsymbol{b}$ is $\leq \boldsymbol{a}$.

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These are examples of $\forall\exists$-statements blocking $\mathcal{P} \subset \mathcal{Q}_0$ but not both $\mathcal{P} \subset \mathcal{Q}_0$ and $\mathcal{P} \subset \mathcal{Q}_1$:

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In 2007, Kent emailed me the following "next" two questions arising from Ahmad's work:

Technical Questions

1. Is there an *Ahmad triple* of Σ_2^0-enumeration degrees, i.e., are there degrees $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ such that $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{b}, \boldsymbol{c})$ form Ahmad pairs?

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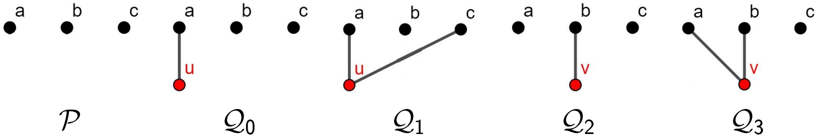
So, e.g., 1 is an example of simultaneously blocking $\mathcal{P} \subset \mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$:



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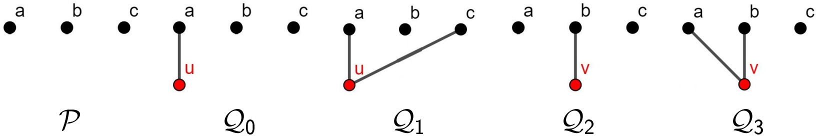

For many years, I believed the answers to both to be "yes".

However, the answer to both questions is “no”:

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Theorem (Kalimullin, Lempp, Ng, Yamaleev, submitted)

There is no cupping Ahmad pair.

The proof turns out to be a non-uniform finite-injury(!) argument.

Given the difficulty of the overall problem of deciding the ∀∃-theory, we are currently concentrating on the following subproblem:

1-Point Extensions of Antichains

Decide, given a finite antichain $\mathcal{P} = \{a_0, \dots, a_n\}$ and 1-point extensions $\mathcal{Q}_S = \{a_0, \dots, a_n, x_S\}$ and $\mathcal{Q}^T = \{a_0, \dots, a_n, x^T\}$ for some nonempty subsets $S, T \subseteq \{0, \dots, n\}$ (where $x_S < a_i$ iff $i \in S$; and $x^T > a_i$ iff $i \in T$),

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Extendible iff there is a singleton T.
We have now found a (complicated) complete characterization for the above subproblem involving only extensions $\mathcal{Q}_S$.
We have no working conjecture that combines the $\mathcal{Q}_S$ and the $\mathcal{Q}^T$.

Main Theorem (Goh, Lempp, Ng, M. Soskova, in preparation)

Fix $n > 0$ and $\mathcal{S} \subseteq \mathcal{P}(\{0, \ldots, n\}) - \{\emptyset\}$.
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Then some embedding of $\mathcal{P}$ into $\mathcal{S}_e$ cannot be extended to an embedding of $\mathcal{Q}_S$ for any $S \in \mathcal{S}$ *("$\mathcal{S}$ can be blocked")* iff $(*)$ holds:

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  - for each $i \in S_0$, $\{i\} \cup \nu(i) \notin \mathcal{S}$, and
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$\{0, \dots, n\} \neq \bigcup \mathcal{S}$: Fix $j \in \{0, \dots, n\} - \bigcup \mathcal{S}$ and make each $\boldsymbol{a}_k$ (for $k \neq j$) form an Ahmad pair with $\boldsymbol{a}_j$.

The most difficult condition of $(*)$ concerns the assignment $\nu : S_0 \to \mathcal{P}(S_1) - \{\emptyset\}$ satisfying

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$\boldsymbol{a}_0$ forms an Ahmad pair with $\boldsymbol{a}_1$;
so $S_0 = \{0\}$ and $\nu : 0 \mapsto \{1\}$, namely, $\mathcal{S} = \{\{0\}\}$ can be blocked, but $\{\{0\}, \{0, 1\}\}$ cannot.

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Harder example showing the second bullet is needed:

$\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ both form an Ahmad pair with $\boldsymbol{a}_2$, and $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ form a minimal pair; so $S_0 = \{0,1\}$ and $\nu : 0, 1 \mapsto \{2\}$,
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The most difficult condition of $(*)$ concerns the assignment $\nu : S_0 \to \mathcal{P}(S_1) - \{\emptyset\}$ satisfying

- for each $i \in S_0$, $\{i\} \cup \nu(i) \notin \mathcal{S}$, and
- for each $F \subseteq S_0$ with $|F| > 1$, we have $\bigcap\{\nu(i) \mid i \in F\} \notin \mathcal{S}$:

Easy example showing the first bullet is needed:

$\boldsymbol{a}_0$ forms an Ahmad pair with $\boldsymbol{a}_1$;
so $S_0 = \{0\}$ and $\nu : 0 \mapsto \{1\}$, namely, $\mathcal{S} = \{\{0\}\}$ can be blocked, but $\{\{0\}, \{0,1\}\}$ cannot.

Harder example showing the second bullet is needed:

$\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ both form an Ahmad pair with $\boldsymbol{a}_2$, and $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ form a minimal pair; so $S_0 = \{0,1\}$ and $\nu : 0,1 \mapsto \{2\}$,
namely, $\mathcal{S} = \{\{0\}, \{1\}, \{0,1,2\}\}$ and even
$\mathcal{S} = \{\{0\}, \{1\}, \{0,1\}, \{0,1,2\}\}$ can be blocked.
But: Note that the first bullet fails for $\mathcal{S} = \{\{0\}, \{1\}, \{0,2\}\}$, so this cannot be blocked.

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Now suppose $\mathcal{P}$ embeds via degrees $\boldsymbol{a}_i$. For each $i \in S_0$, fix nonzero $\boldsymbol{v}_i < \boldsymbol{a}_i$ with $\boldsymbol{v}_i \not\leq \boldsymbol{a}_k$ for all $k \in S_0 - \{i\}$, and set $\nu(i) = \{j \in S_1 \mid \boldsymbol{v}_i \leq \boldsymbol{a}_j\}$, so $\{i\} \cup \nu(i) \notin \mathcal{S}$ (and $\nu(i) \neq \emptyset$).

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So ν is an assignment as desired.

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$$\mathcal{I}_j : A_j \neq W \qquad (\text{if } j \in S_1 - \bigcup_{i \in S_0} \nu(i))$$

Thanks!