Wadge Degrees, Games, and the Separation and Reduction Properties

Daniel Turetsky
Victoria University of Wellington

Joint work with Noam Greenberg
Partly joint with Matthew Harrison-Trainor

October 2022

Wadge reducibility

Pick a Polish space X – usually ω^ω or 2^ω.

For $A, B \subseteq X$, write $A \leq_W B$ if there is a continuous f with $A = f^{-1}(B)$.

Equivalently, $\forall x \in X$, $x \in A \iff f(x) \in B$.

Analogy: $A, B \subseteq \omega$, $A \leq_m B$.

Semi-linear ordering

Lemma (Wadge)

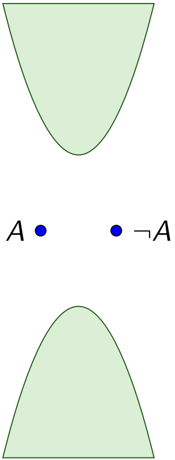
For any Borel $A, B \subseteq X$, at least one of these holds:

- $A \leq_W B$;
- $B \leq_W \neg A$.

Corollary

For any Borel $A, B \subseteq X$:

- $B \equiv_W A$ or $B \equiv_W \neg A$; or
- $B <_W A$ and $B <_W \neg A$; or
- $B >_W A$ and $B >_W \neg A$.

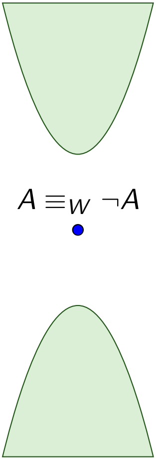


A self-dual degree

Let $A = [0] = \{0^\frown x : x \in \omega^\omega\}$.

$$f(b^\frown x) = \begin{cases} 1^\frown x & \text{if } b = 0, \\ 0^\frown x & \text{if } b > 0. \end{cases}$$

f shows $A \equiv_W \neg A$.



Wadge classes

$\boldsymbol{\Gamma} \subseteq \mathcal{P}(X)$ is a *Wadge class* if it is downwards closed under $\leq_W$.

$\boldsymbol{\Gamma}$ is a *pointclass* if there is $A \in \boldsymbol{\Gamma}$ with $\boldsymbol{\Gamma} = \{B : B \leq_W A\}$.

The ordering becomes containment: if $\boldsymbol{\Gamma}_i$ is the pointclass of A_i, then $A_0 \leq_W A_1 \Leftrightarrow \boldsymbol{\Gamma}_0 \subseteq \boldsymbol{\Gamma}_1$.

Some example pointclasses: $\boldsymbol{\Sigma}^0_1, \boldsymbol{\Pi}^0_1, \boldsymbol{\Sigma}^0_2, \boldsymbol{\Pi}^0_2, \ldots$.

Duals and deltas

For $\boldsymbol{\Gamma}$ a Wadge class:

- The *dual* of $\boldsymbol{\Gamma}$ is $\check{\boldsymbol{\Gamma}} = \{\neg B : B \in \boldsymbol{\Gamma}\}$. If $\boldsymbol{\Gamma}$ is a pointclass, so is $\check{\boldsymbol{\Gamma}}$.
- $\boldsymbol{\Gamma}$ is *self-dual* if $\boldsymbol{\Gamma} = \check{\boldsymbol{\Gamma}}$.
- The *ambiguous class* of $\boldsymbol{\Gamma}$ is $\Delta(\boldsymbol{\Gamma}) = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$.

$\check{\boldsymbol{\Sigma}}^0_\alpha = \boldsymbol{\Pi}^0_\alpha$.

The class of clopen sets is self-dual.

$\Delta(\boldsymbol{\Sigma}^0_\alpha) = \Delta(\boldsymbol{\Pi}^0_\alpha) = \boldsymbol{\Delta}^0_\alpha$.

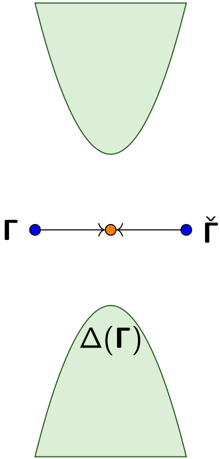
Well ordering

By Wadge's lemma, the pointclasses of Borel sets are totally-ordered, if we collapse a class with its dual.

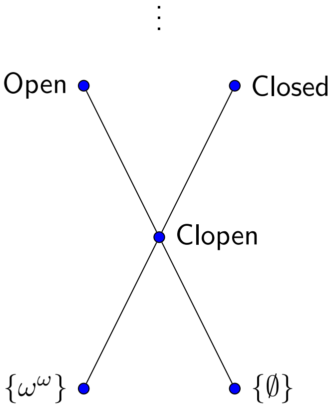
Theorem (Martin)

This is a well-ordering.

The order-type is beyond ω_1 — needs Veblen functions.



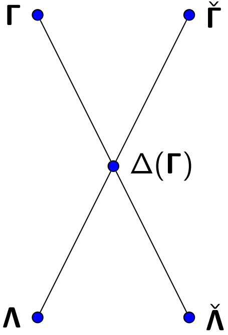
The start of the order



Successor steps

If $\mathbf{\Lambda}$ is non-self-dual, there is non-self-dual $\mathbf{\Gamma}$ so that this picture is convex.

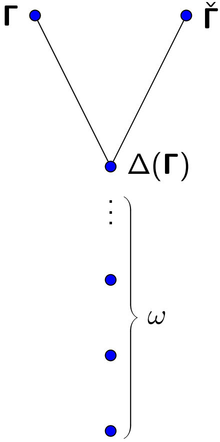
All are pointclasses.



Countable limits

At a countable cofinality limit point, there is non-self-dual $\boldsymbol{\Gamma}$ so that $\Delta(\boldsymbol{\Gamma})$ is the limit.

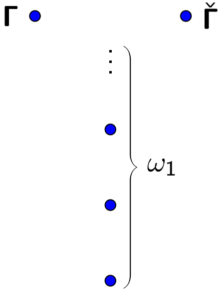
All are pointclasses.



Uncountable limits

At an uncountable cofinality limit point, there is non-self-dual $\mathbf{\Gamma}$ so that $\Delta(\mathbf{\Gamma})$ is the union of the earlier classes.

$\Delta(\mathbf{\Gamma})$ is *not* a pointclass.



A note about subscripts

For $n < \omega$, Σ^0_{n+1} means enumerable with n jumps.

For $\alpha \geq \omega$, Σ^0_α means enumerable with α jumps.

These can be unified:

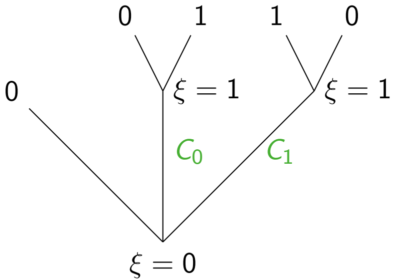
- For $n < \omega$, $1 + n = n + 1$. $\quad \Sigma^0_{1+n} = \Sigma^0_{n+1}$.
- For $\alpha \geq \omega$, $1 + \alpha = \alpha$. $\quad \Sigma^0_{1+\alpha} = \Sigma^0_\alpha$.

For all α, $\Sigma^0_{1+\alpha}$ means enumerable with α jumps.

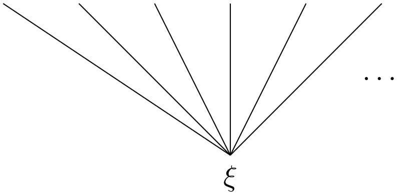
An example description

BiSep($\mathbf{\Sigma}^0_1, \mathbf{\Sigma}^0_2$) is the class of all sets $(C_0 \cap A_0) \cup (C_1 \cap A_1)$ satisfying:

- $C_0, C_1 \in \mathbf{\Sigma}^0_1$;
- $C_0 \cap C_1 = \emptyset$;
- $A_0 \in \mathbf{\Sigma}^0_2$;
- $A_1 \in \check{\mathbf{\Sigma}}^0_2 = \mathbf{\Pi}^0_2$.



Changing children



On a $z \in \omega^\omega$, a computable process with access to $z^{(\xi)}$ starts at the leftmost child, and can move to another child in a c.e. way.

Another example

$D^\eta(\mathbf{\Sigma}^0_{1+\xi})$ is the η-level in the Hausdorff difference hierarchy. Members are sets of the form

$$\bigcup_{\substack{\alpha<\eta \\ \text{parity}(\alpha)\neq\text{parity}(\eta)}} \left(A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta \right)$$

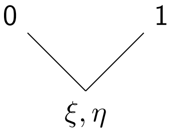
where the A_α are all $\mathbf{\Sigma}^0_{1+\xi}$.

How I understand a $D^\eta(\mathbf{\Sigma}^0_{1+\xi})$ set

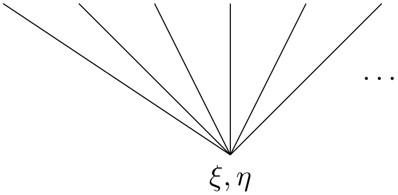
From $z^{(\xi)}$, uniformly compute two sequences $(a_n)_{n\in\omega}$ and $(\beta_n)_{n\in\omega}$ satisfying:

- $a_0 = 0$ and $\beta_0 = \eta$;
- For all n, $\beta_n \geq \beta_{n+1}$;
- If $a_n \neq a_{n+1}$, then $\beta_n > \beta_{n+1}$; and
- $\lim_n a_n$ indicates if z belongs to the set.

Call this an η-*bounded approximation*.



Changing children more

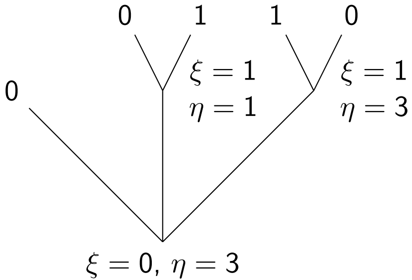


On a $z \in \omega^\omega$, a computable process with access to $z^{(\xi)}$ starts at the leftmost child, and can move to other children in an η-bounded approximation.

The descriptions

A description is a labeled tree $T \subseteq \omega^\omega$ such that:

- T is well-founded;
- Each leaf of T is labeled with 0 or 1;
- Each non-leaf $\sigma \in T$ is labeled with ordinals ξ_σ and η_σ;
- If $\sigma \subseteq \tau \in T$ are not leaves, then $\xi_\sigma \leq \xi_\tau$.



Names for sets

If Γ is a description, a Γ-*name for a set* consists of a collection $\{\Phi_\sigma : \text{non-leaf } \sigma \in T_\Gamma\}$. Each Φ_σ is a functional for generating an η_σ-bounded approximation to a child of σ, where the starting value is the leftmost child. (All relative to some real parameter.)

For each $z \in \omega^\omega$, build a path through T_Γ:

- Start at the root.
- If we've reached a non-leaf $\sigma \in T_\Gamma$, we choose the child $\lim \Phi_\sigma(z^{(\xi_\sigma)})$.

Membership of z is determined by the label on the leaf we reach: 0 or 1.

Described classes

If Γ is a description, let $\mathbf{\Gamma}$ be the collection of all sets which have a Γ-name.

Theorem

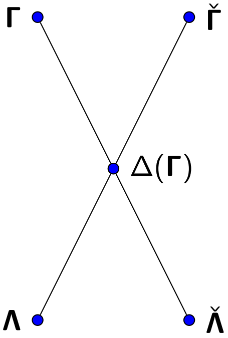
The $\mathbf{\Gamma}$ are precisely the non-self dual Borel pointclasses.

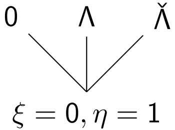
Any self-dual Borel pointclass is $\Delta(\mathbf{\Gamma})$ for some description Γ.

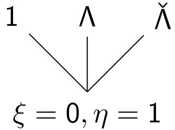
A description of $\check{\mathbf{\Gamma}}$ can be made from Γ by swapping all 0s and 1s at the leaves.

Returning to successors

For any description Λ, we get a description for the next pair up:

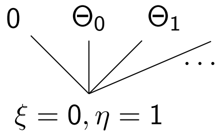


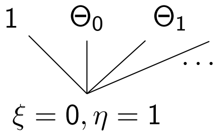


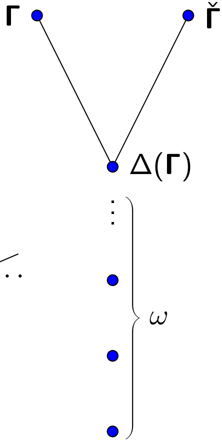


Returning to countable limits

For any increasing sequence $\Theta_0, \Theta_1, \ldots,$ we get a description for pair at the limit:







Returning to the uncountable limits

Theorem

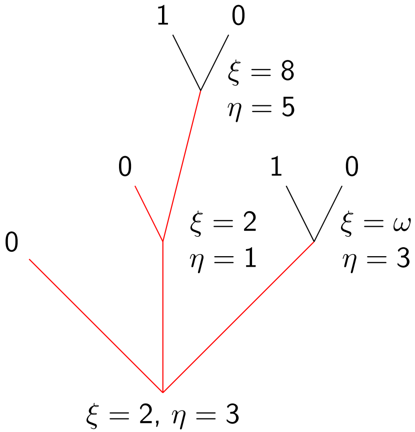
$(\mathbf{\Gamma}, \check{\mathbf{\Gamma}})$ *is a limit of uncountable cofinality precisely when there is a* σ *along the leftmost branch of* Γ *with* $\xi_\sigma > 0$.

A game for containment – $GC(\Gamma, \Lambda)$

Γ and Λ are descriptions.

Consider a convex piece of T_Γ or T_Λ in which all internal nodes have the same ξ.

The piece is a tree within T_Γ or T_Λ, not necessarily with the same root.

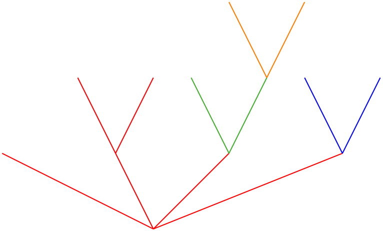


A game for containment – $GC(\Gamma, \Lambda)$

Divide T_Γ and T_Λ into maximal pieces like this.

Each leaf of a piece is either a leaf of the original tree or the root of another piece.

We have a Γ-player and Λ-player. The players will move from piece to piece on their respective trees, moving towards a leaf.



A game for containment – $GC(\Gamma, \Lambda)$

Now compare ξ^Γ and ξ^Λ, the ξ of the current two pieces. If one player has reached a leaf of the original tree, treat their ξ as ∞.

If $\xi^\Gamma < \xi^\Lambda$, the Γ player picks a leaf of their piece and moves there. The Λ player stays where they are.

If $\xi^\Gamma > \xi^\Lambda$, the Λ player picks a leaf of their piece and moves there. The Γ player stays where they are.

If $\xi^\Gamma = \xi^\Lambda < \infty$, things are more complicated.

$GC(\Gamma, \Lambda)$ – when $\xi^\Gamma = \xi^\Lambda$

The players dynamically choose leaves of their current pieces.

Each internal node σ in a piece begins pointing towards its leftmost child.

On their turn, a player may change some nodes to point to other children, or they may pass. The changes of σ must be η_σ-bounded.

This process ends when the players pass around. Each player starts at the root of their piece and follows the pointers to reach a leaf. This is their next piece.

$GC(\Gamma, \Lambda)$ – victory

The game ends when both players reach a leaf of their original tree.

If the leaves have the same label (0 or 1), the Λ-player wins.
Otherwise, the Γ-player wins.

Theorem

The Λ-player has a winning strategy for $GC(\Gamma, \Lambda)$ iff $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$.

Monotone sequences and descriptions

A sequence of descriptions (Θ_n) is *monotone* if for all n, $\check{\boldsymbol{\Theta}}_n \subseteq \boldsymbol{\Theta}_{n+1}$.

Equivalently, $\boldsymbol{\Theta}_{n+1} = \check{\boldsymbol{\Theta}}_n$ or $\boldsymbol{\Theta}_{n+1} \supset \check{\boldsymbol{\Theta}}_n$.

A description is monotone if for every non-leaf σ, the subdescriptions rooted at σ's children form a monotone sequence.

Every non-self-dual Borel pointclass has a monotone description. Henceforth all descriptions are monotone.

Distinguishing the halves of a pair

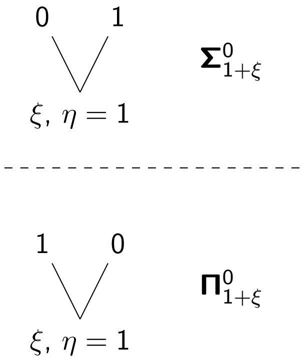
Definition

A description has Σ-*type* if its leftmost leaf is labeled 0.

Otherwise, it has Π-*type*.

Lemma

The type of a class is invariant.



The separation property

$\boldsymbol{\Pi}^0_{1+\xi}$ has the *separation property*: for any two disjoint sets in $\boldsymbol{\Pi}^0_{1+\xi}$, there is a $\Delta(\boldsymbol{\Pi}^0_{1+\xi}) = \boldsymbol{\Delta}^0_{1+\xi}$ separator.
(This also works for lightface classes on ω.)

Theorem

$\boldsymbol{\Gamma}$ *has the separation property iff* Γ *is of* Π*-type.*

The reduction property

$\mathbf{\Sigma}^0_{1+\xi}$ has the *reduction property*: for any $A_0, A_1 \in \mathbf{\Sigma}^0_{1+\xi}$, there are $B_0, B_1 \in \mathbf{\Sigma}^0_{1+\xi}$ satisfying:

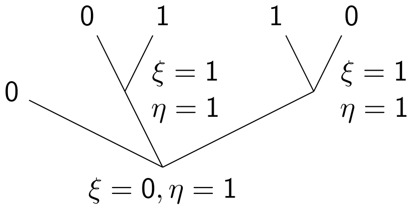
- $B_0 \subseteq A_0$ and $B_1 \subseteq A_1$;
- $B_0 \cap B_1 = \emptyset$; and
- $B_0 \cup B_1 = A_0 \cup A_1$.

(This also works for lightface classes on ω.)

The situation is more complex

Example

BiSep($\mathbf{\Sigma}^0_1, \mathbf{\Sigma}^0_2$) does not have the reduction property



Lemma

If $\mathbf{\Gamma}$ *has* Σ*-type and is closed under finite intersections, then it has the reduction property.*

Example

$D^\eta(\mathbf{\Sigma}^0_{1+\xi})$ has the reduction property but is not closed under finite intersections (when $\eta > 1$).

More than Σ-type

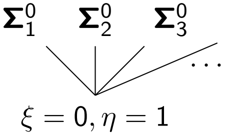
Definition

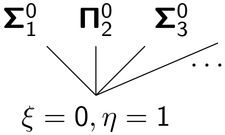
A description has *hereditarily* Σ*-type* if for every internal node, the leftmost leaf extending that node is labeled 0.

Theorem

$\mathbf{\Gamma}$ *has the reduction property iff it has a description of hereditarily* Σ*-type.*

Unfortunately, hereditarily Σ-type is not invariant.





Generalizing separation and reduction

Definition

$\mathbf{\Lambda}$ *separates* $\mathbf{\Gamma}$ if for any two disjoint sets in $\mathbf{\Gamma}$, there is a separator in $\Delta(\mathbf{\Lambda})$.

$\mathbf{\Lambda}$ *reduces* $\mathbf{\Gamma}$ if for any $A_0, A_1 \in \mathbf{\Gamma}$, there are $B_0, B_1 \in \mathbf{\Lambda}$ satisfying:

- $B_0 \subseteq A_0$ and $B_1 \subseteq A_1$;
- $B_0 \cap B_1 = \emptyset$; and
- $B_0 \cup B_1 = A_0 \cup A_1$.

Separation and reduction games

Define games $GS(\Gamma,\Lambda)$ and $GR(\Gamma,\Lambda)$.

Like $GC(\Gamma,\Lambda)$, but each player has two copies of their tree: T^a_Γ, T^b_Γ, T^a_Λ and T^b_Λ. They work their way down both simultaneously.

Each player ends up with two labels, one from each tree.

When the Λ-player wins $GS(\Gamma,\Lambda)$

Γ-player	Λ-player
$(1,1)$	Anything
$(1,0)$	$(1,0)$
$(0,1)$	$(0,1)$
$(0,0)$	$(0,1)$ or $(1,0)$

When the Λ-player wins $GR(\Gamma,\Lambda)$

Γ-player	Λ-player
$(0,0)$	$(0,0)$
$(0,1)$	$(0,1)$
$(1,0)$	$(1,0)$
$(1,1)$	$(1,0)$ or $(0,1)$

What the games tell us

Theorem

The Λ*-player has a winning strategy for* $GS(\Gamma, \Lambda)$ *iff* Λ *separates* Γ*.*

The Λ*-player has a winning strategy for* $GR(\Gamma, \Lambda)$ *iff* Λ *reduces* Γ*.*

Back to computability theory

These descriptions can also be used on ω to get m-reduciblity classes.

Now z is a number, so instead of $z^{(\xi)}$, just use $\emptyset^{(\xi)} \oplus \{z\}$. Otherwise, nothing changes about names and descriptions.

When the tree and the ordinals are all finite, we get precisely Selivanov's hierarchy of Boolean terms on Σ^0_{1+n}-sets. Semi-linear ordering is immediate: GC is a finite game, so it has a computable winning strategy.

Our student Qi Renrui is working on this to extend Selivanov's hierarchy beyond arithmetical.

Thank you.