

Transfer of Ramsey phenomena via semi-retractions

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Impacts of the pandemic



Ramsey's theorem

Finite Ramsey's theorem

For every natural numbers $k \leq l$ and a number of colours r , there is a number N such that for every colouring $c : [N]^k \rightarrow \{0, 1, \dots, r-1\}$, there is a subset Y of N of size l such that $[Y]^k$ is monochromatic.

Ramsey's theorem

Let k be a natural number and let r be a number of colours. For every colouring $c : [\mathbb{N}]^k \rightarrow \{0, 1, \dots, r-1\}$ there is an infinite subset Y of \mathbb{N} such that $[Y]^k$ is monochromatic.

Ramsey degrees

Let \mathcal{K} be a class of finitely generated structures in a language \mathcal{L} . Let $A, B \in \mathcal{K}$. The set of embeddings of A in B is denoted by

$$\text{Emb}(A, B)$$

Definition

$A \in \mathcal{K}$ has a **finite Ramsey degree** in \mathcal{K} if there is t_A such that for every $B \in \mathcal{K}$ and number of colours r there is $C \in \mathcal{K}$ such that for every **colouring** $c : \text{Emb}(A, C) \rightarrow \{0, 1, \dots, r-1\}$ there is $\phi \in \text{Emb}(B, C)$ such that $\phi \circ \text{Emb}(A, B)$ is t_A -chromatic.

CLASSES WITH FINITE RAMSEY DEGREES:

- 1 finite sets (Ramsey);
- 2 finite linear orders (Ramsey);
- 3 finite Boolean algebras (Graham and Rothschild);
- 4 finite (K_n -free) graphs (Nešetřil – Rödl; Abramson – Harrington).

Semi-retractions

\mathcal{A}, \mathcal{B} – structures (typically not in the same language)

Definition

An injection $h : \mathcal{A} \rightarrow \mathcal{B}$ is **quantifier-free type-respecting** (qftpr) if whenever n -tuples \bar{a}, \bar{b} from \mathcal{A} have the same quantifier-free type, then so do $h(\bar{a}), h(\bar{b})$.

Definition (Scow)

A pair (g, f) is a **semi-retraction between \mathcal{A} and \mathcal{B}** if $g : \mathcal{A} \rightarrow \mathcal{B}$ and $f : \mathcal{B} \rightarrow \mathcal{A}$ are qftpr injections and $f \circ g : \mathcal{A} \rightarrow \mathcal{A}$ is an embedding.

$\mathcal{A} = (\mathbb{Q}, <)$, $\mathcal{R}_{<}$ is the random ordered graph

$g : \mathbb{Q} \rightarrow \mathcal{R}_{<}$ injects \mathbb{Q} to a complete subgraph of \mathcal{R} in an order preserving way ($\mathcal{R}_{<}$ is universal for countable ordered graphs).

$f : \mathcal{R} \rightarrow \mathbb{Q}$ forgets the linear order, in other words, it is the reduct to the language containing only the graph relation

$f \circ g$ is an embedding

(\mathcal{R}, E) is the random graph, \mathcal{B} is the countable atomless Boolean algebra

Extend \mathcal{B} by graph structure: (a, b) is an edge iff $a \wedge b \neq \mathbf{0}$.

$f : \mathcal{B} \rightarrow \mathcal{R}$ is any graph embedding (\mathcal{R} is countably universal).

$g : \mathcal{R} \rightarrow \mathcal{B}$: Let $B = (b_i)_{i \in \omega}$ be an infinite antichain in \mathcal{B} . Let $B^i = (b_i^j)_{j \in \omega}$ be an infinite antichain below b_i .

Let $\{v_i : i \in \omega\}$ be any enumeration of vertices of \mathcal{R} .

$$g(v_i) = b_i \vee \bigvee_{j < i} b_i^j(\varepsilon),$$

where

$$b_i^j(\varepsilon) = \begin{cases} b_i^j & \text{if } (v_i, v_j) \in E \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Why do we define g like that?

In \mathcal{B} , there are different quantifier-free types of edges:

(a, b) with $a < b$ or $b < a$

(a, b) with $a \wedge b \neq \mathbf{0}$ and $a \wedge \neg b \neq \mathbf{0} \neq \neg a \wedge b$.

Clearly, we need to choose the second type. But even then, such edges will form different types of triangles ... But in our definition for every v_i, v_j, v_k , we have $g(v_i) \wedge g(v_j) \wedge g(v_k) = \mathbf{0}$.

So we can use the full disjunctive normal form to show that a qf type of n -tuple in the image of g is completely determined by the graph structure.

(\mathcal{H}_n, E) is the random n -regular hypergraph, \mathcal{B} is the countable atomless Boolean algebra

We extend \mathcal{B} by n -regular hypergraph relation: (b_0, \dots, b_{n-1}) is a hyperedge if $\bigwedge_{i=0}^{n-1} b_i \neq \mathbf{0}$.

$f : \mathcal{B} \rightarrow \mathcal{H}_n$ is any hypergraph embedding.

$g : \mathcal{H}_n \rightarrow \mathcal{B}$: Let $B_k = \{b_{\vec{i}} : \vec{i} \in \omega^{[k]}\}$ be antichain for $1 \leq k \leq n$ s.t. $\vec{i} \sqsubseteq \vec{j} \rightarrow b_{\vec{j}} < b_{\vec{i}}$.

Let $\{v_l : l \in \omega\}$ be an enumeration of vertices of \mathcal{H}_n .

$$g(v_l) = \bigvee_{\vec{i}(|\vec{i}|-1)=l, |\vec{i}|<n} b_{\vec{i}} \vee \bigvee_{\vec{i}(|\vec{i}|-1)=l, |\vec{i}|=n, (v_{i(0)}, \dots, v_{i(n-1)}) \in E} b_{\vec{i}}.$$

$(\mathcal{R}, \mathcal{B})$ was inspired by Mašulović's transfer of the Ramsey property by a category-theoretic notion of pre-adjunction. He noted that his method did not immediately generalize to hypergraphs as in $(\mathcal{H}_n, \mathcal{B})$.

When semi-retractions transfer finite small Ramsey degrees

$\text{Age}(\mathcal{A}) =$ class of finitely-generated substructures of \mathcal{A} .

Theorem

Let \mathcal{A} be a locally finite semi-retract of \mathcal{B} via (g, f) . Let $A \in \text{Age}(\mathcal{A})$. If $\langle g(A) \rangle$ has finite Ramsey degree in $\text{Age}(\mathcal{B})$ then so does A in $\text{Age}(\mathcal{A})$.

Proof.

$A, B \in \text{Age}(\mathcal{A})$, $c : \text{Emb}(A, \mathcal{A}) \rightarrow \{0, 1, \dots, r-1\}$. WLOG there are $A_0, B_0 \in \text{Age}(\mathcal{A})$ with $A = fg(A_0)$, $B = fg(B_0)$.

$A' = \langle g(A) \rangle_{\mathcal{B}}$, $B' = \langle g(B) \rangle_{\mathcal{B}}$.

$c' : \text{Emb}(A', \mathcal{B}) \rightarrow \{0, 1, \dots, r-1\}$, $c'(e) = c(f(g(e \upharpoonright A_0)))$.
 $h \in \text{Emb}(B', \mathcal{B})$ with $c \upharpoonright h \circ \text{Emb}(A', B') \leq t_{A'}$ -chromatic.

$f(h \upharpoonright g(B_0))$ works.



G – topological group, X – compact Hausdorff space

G -flow is a continuous action of G on X

G -flow on X is **minimal** if X has no proper non-empty closed G -invariant subset

Universal minimal G -flow, $M(G)$, has every minimal G -flow as its quotient.

EXAMPLES

- 1 $M(\text{Aut}(\mathbb{Q}, <))$ is trivial.
- 2 $M(S_\infty(\mathbb{N}))$ is $\text{LO}(\mathbb{N})$.
- 3 $M(\text{Homeo}_+(\mathbb{S}^1))$ is \mathbb{S}^1 .
- 4 $M(\mathbb{Z})$ is huge and yet to be fully understood.

\mathcal{A} is **ultrahomogeneous** if every finite partial self-isomorphism extends to a full automorphism.

If \mathcal{A} is countable ultrahomogeneous then $\text{Age}(\mathcal{A})$ has finite Ramsey degrees iff $\text{Aut}(\mathcal{A})$ has metrizable universal minimal flow (by Kechris-Pestov-Todorčević and Zucker).

If a countable ultrahomogeneous structure \mathcal{A} is a semi-retract of a countable ultrahomogeneous structure \mathcal{B} and $M(\text{Aut}(\mathcal{B}))$ is metrizable, then $M(\text{Aut}(\mathcal{A}))$ is metrizable.

Question

How are automorphism groups of \mathcal{A} and \mathcal{B} related as topological groups?

Pre-adjunctions

Let \mathbf{C} and \mathbf{D} be categories and let $F : \text{Obj}(\mathbf{D}) \rightarrow \text{Obj}(\mathbf{C})$ and $G : \text{Obj}(\mathbf{C}) \rightarrow \text{Obj}(\mathbf{D})$ be maps on objects. We say that (F, G) is a **pre-adjunction** if for every $A \in \text{Obj}(\mathbf{D})$ and $C \in \text{Obj}(\mathbf{C})$ we have a map

$$\Phi_{A,C} : \text{hom}_{\mathbf{C}}(F(A), C) \rightarrow \text{hom}_{\mathbf{D}}(A, G(C)),$$

such that

$$\forall A, B \in \text{Obj}(\mathbf{D}) \forall C \in \text{Obj}(\mathbf{C}) \forall v \in \text{hom}_{\mathbf{D}}(A, B) \forall \psi \in \text{hom}_{\mathbf{C}}(F(B), C)$$

$$\exists w \in \text{hom}_{\mathbf{C}}(F(A), F(B)) \text{ such that } \Phi_{A,C}(\psi \circ w) = \Phi_{B,C}(\psi) \circ v.$$

Semi-retractions are pre-adjunctions

Theorem

Any semi-retraction (g, f) between \mathcal{A} and \mathcal{B} defines a pre-adjunction between the categories of finite tuples of \mathcal{A} and \mathcal{B} , respectively, with qftp-preserving injections.

Theorem

Let \mathcal{A} and \mathcal{B} be locally finite and let (g, f) be a semi-retraction between \mathcal{A} and \mathcal{B} . Then there is a pre-adjunction between $\text{Age}(\mathcal{A})$ and $\text{Age}(\mathcal{B})$ with embeddings as morphisms.

Theorem

Suppose that \mathcal{A} and \mathcal{B} are countable structures in relational languages. Let \mathbf{D} be $\text{Age}(\mathcal{A})$ with embeddings as morphisms and let \mathbf{C} be $\text{Age}(\mathcal{B})$ with embeddings as morphisms.

Suppose that (F, G) is a pre-adjunction between \mathbf{D} and \mathbf{C} and assume that F and G preserve cardinality.

Then there is a semi-retraction between \mathcal{A} and a structure \mathcal{B}' whose age is contained in \mathbf{C} . Moreover, \mathcal{A} is a reduct of \mathcal{B}' .

Question

Is every pre-adjunction on ages of countable ultrahomogeneous structures witnessed by a semi-retraction?

Question

Can semi-retractions witness transfer of finite big Ramsey degrees as well?

The end ... to be continued.

THANK YOU!