Effectivizing the theory of Borel equivalence relations

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Today's menu

- Opening
  - Borel and computable reductions
- Developing a computable analog of the Borel theory
  - Dichotomies
  - Orbit equivalence relations
  - Isomorphism relations

Reductions between equivalence relations

A reduction of an equivalence relation E on X to an equivalence relation F on Y is a function $f : X \to Y$ such that

$$x \, E \, y \Leftrightarrow f(x) \, F \, f(y).$$

That is, f pushes down to an injective map on the quotient spaces, $X/_E \to Y/_F$.

By the Axiom of Choice, it suffices that the E-classes are no more than the F-classes to conclude that E reduces to F. Things become much more interesting if we impose definitional or algorithmic requirements on the spaces and functions. In the literature, there are two main interpretations for this reducibility.

Borel reductions

Borel reducibility, denoted $\leq_B$, is defined by assuming that X and Y are Polish spaces and f is Borel. If E and F are Borel bi-reducible, we write $E \sim_B F$.

Borel reducibility was defined, independently (but with the exact same notation and terminology),

- by **H. Friedman** and **Stanley** (1989), with the goal of evaluating the complexity of familiar isomorphism relations
- and by **Harrington**, **Kechris**, and **Louveau** (1990), with the goal of extending the **Glimm-Effros** dichotomy to arbitrary Borel equivalence relations.

Since then, Borel reductions have been widely explored, showing deep connections with topology, group theory, combinatorics, model theory, and ergodic theory – to name a few.

Computable reductions

Computable reducibility, denoted $\leq_c$, is defined by assuming that X and Y coincide with ω (or sometimes that $X, Y \subseteq \omega$) and f is computable. If E and F are computably bi-reducible, we write $E \sim_c F$.

The history of $\leq_c$ is intricate. Despite being introduced in the 1970s (hence, even *before* $\leq_B$) and being examined, among others, by **Ershov** in the East and **Lachlan** in the West, it was forgotten and rediscovered multiple times, often reappearing under a different name.

Computable reductions found remarkable applications in various fields, including the theory of numberings, proof theory, computable structure theory, combinatorial algebra, and theoretical computer science. But a systematic study of $\leq_c$ has really begun to take off only recently.

Developing a computable analog of the Borel theory

Silver's dichotomy

The simplest Borel equivalence relations are the identities. It is immediate to see that $\mathrm{Id}(\omega) <_B \mathrm{Id}(2^\omega)$. The next classic result implies that $\mathrm{Id}(2^\omega)$ reduces to any Borel (in fact, to any co-analytic) equivalence relation with uncountably many classes.

Silver's dichotomy:

Let E be a Borel equivalence relation on a standard Borel space. Then, exactly one the following holds:

1. $E \leq_B \mathrm{Id}(\omega)$,
2. $\mathrm{Id}(2^\omega) \leq_B E$.

So, $\mathrm{Id}(2^\omega)$ is the successor to $\mathrm{Id}(\omega)$ in the Borel hierarchy. Is there a successor to $\mathrm{Id}(2^\omega)$? Strikingly, there is.

A glance to E_0 and beyond, I

Glimm-Effros dichotomy (**Harrington, Kechris, Louveau**):

Denote by E_0 the relation of eventual agreement on 2^ω. Let E be a Borel equivalence relation on a standard Borel space. Then, exactly one of the following holds:

1. $E \leq_B \mathrm{Id}(2^\omega)$,
2. $E_0 \leq_B E$.

Beyond E_0, the landscape is much wilder. Say that a E is a node if it is $\leq_B$-comparable with any Borel equivalence relation:

- *No Borel equivalence relation $E >_B E_0$ is a node.*
  Kechris, Louveau (1997)
- *There is an embedding from $\langle \mathcal{P}(\omega), \subseteq^* \rangle$ into the Borel hierarchy.*
  Louveau, Velickovic (1994)

A glance to E_0 and beyond, II

Yet, *local* dichotomies still emerge. Let E_1 be the relation of eventual agreement on sequences of reals.

Theorem (Kechris, Louveau)

E_1 is minimal above E_0. In fact, let $E \leq_B E_1$. Then, exactly one of the following holds:

1. $E \leq_B E_0$,
2. $E_1 \leq_B E$.

From Borel to computable, I

Let us now move to the computable setting, with the goal of building an effective counterpart to the Borel theory. In the (standard) computable setting, all equivalence relations are to be defined on ω. This is not an issue for $\mathsf{Id}(\omega)$. But how to deal with, e.g., $\mathsf{Id}(2^\omega)$ or E_0?

Following Coskey, Hamkins, and R. Miller (2012), we adapt benchmark relations from the Borel theory by restricting them to the c.e. sets. This naturally give rise to equivalence relations on the natural numbers. Indeed, if E is on the c.e. sets, then we let, for all $e, i \in \omega$,

$$e \, E^{ce} \, i \Leftrightarrow W_e \, E \, W_i.$$

From Borel to computable, II

So, $\mathsf{Id}(2^\omega)$ translates to the equality of c.e. sets, given by

$$e =^{ce} i \Leftrightarrow W_e = W_i.$$

Similarly, we let

$$e\, E_0^{ce}\, i \Leftrightarrow W_e \bigtriangleup W_i \text{ is finite.}$$

E_1^{ce} is defined by regarding at c.e. sets as subsets of $\omega \times \omega$. Formally, let $W_e^{[n]} := \{\langle x, n\rangle \in W_e : x \in \omega\}$ be the nth column of W_e. Then,

$$e\, E_1^{ce}\, i \Leftrightarrow (\forall^\infty n)(W_e^{[n]} = W_i^{[n]}).$$

Reductions between E^{ce}'s, I

Theorem (Coskey, Hamkins, R. Miller)

$\mathrm{Id}(\omega) <_c =^{ce} <_c E_0^{ce}$.

Proof idea: The reductions closely resemble the Borel ones. Nonreductions are far easier to get than in the Borel framework. Calculating the complexity of the relations involved (as set of pairs) suffices:

- $\mathrm{Id}(\omega)$ is Δ^0_1,
- $=^{ce}$ is Π^0_2,
- E_0^{ce} is Σ^0_3.

Reductions between E^{ce}'s, II

Theorem (Coskey, Hamkins, R. Miller)

$E_0^{ce} \sim_c E_1^{ce}$.

That E_1^{ce} reduces to E_0^{ce} is surprising and it breaks with the Borel theory. In fact, it turns out that E_0^{ce} is as complex as possible:

Theorem (Ianovski, R. Miller, Ng, Nies)

E_0^{ce} is Σ_3^0 universal.

Silver's dichotomy fails for computable reducibility

There is no analog of **Silver's dichotomy** for $\leq_c$. For all $e, i \in \omega$, define

- $e \, E_{\min} \, i \Leftrightarrow (\min W_e = \min W_i)$,
- $e \, E_{\max} \, i \Leftrightarrow (\max W_e = \max W_i$ or $|W_e| = |W_i| = \infty)$.

Theorem (Coskey, Hamkins, R. Miller)

$E_{\min}$ and $E_{\max}$ are c-incomparable and they both reduce to $=^{ce}$.

Other dichotomies fail as well (stay tuned). However, the failure of dichotomies is to be expected: first, contrary to $\leq_B$, computable reducibility is sensible to the complexity of relations/classes involved; secondly, controlling fixed points given by the recursion theorem is a formidable tool for diagonalizing.

Orbit equivalence relations

Countable Borel equivalence relations

A fundamental subclass of Borel equivalence relations, named countable Borel equivalence relations (cbers), consists of those with countable equivalence classes. This study is intertwined with that of the equivalence relations which can be realized by Borel actions of countable groups.

Let G be a group acting on a standard Borel space. Then the orbit equivalence relation E_G is given by

$$x \, E_G \, y \Leftrightarrow (\exists \gamma \in G)(\gamma \cdot x = y).$$

Group actions

For example,

- The action of $\mathbb{Z}$ on 2^ω induced by the odometer map (i.e., $+1 \bmod 2$ with right carry) produces an equivalence relation which almost coincides with E_0, but it glues $[1^\infty]_{E_0}$ with $[0^\infty]_{E_0}$.
- For each countable group G, the shift action of G on the space 2^G is given by

$$(g \cdot p)_h = p_{g^{-1}h},$$

for $g, h \in G$ and $p \in 2^G$. (If $G = \mathbb{Z}$, this corresponds to left shift of doubly-infinite binary sequences).

Realizing cbers by group actions

Theorem (Feldman, Moore)

If E is a cber on a standard Borel space X, then there is a countable group G and a Borel action of G on X such that $E = E_G$.

The proof relies on **Luzin-Novikov Uniformization**, which ensures that every countable Borel equivalence relation has a uniform Borel enumeration of each class.

The hierarchy of cbers is rich and complicated. However, it has a top element. Denote by E_∞ the shift action $\mathbb{F}_2$ (the free group with 2-generators) on $2^{\mathbb{F}_2}$.

Theorem (Dougherty, Jackson, Kechris)

E_∞ is a universal cber *(that is, $E \leq_B E_\infty$ for all cbers E).*

Orbit equivalence relations under computable lenses

Denote by **CE**, the collection of c.e. sets (to be understood *extensionally*, i.e., as just subsets of ω).

Coskey, Hamkins, R. Miller (2012):

- The action of a computable group G acting on **CE** is computable in indices if there is computable α so that

$$W_{\alpha(g,e)} = g \cdot W_e.$$

  The induced orbit equivalence relation is denoted E_G^{ce}.
- E^{ce} is enumerable in indices if there is computable α so that, for all $i \in \omega$,

$$e\, E^{ce}\, i \Leftrightarrow (\exists n)(W_{\alpha(e,n)} = W_i).$$

Realizing E_0^{ce} via a group action, I

Is there an effective analog of Feldman-Moore? That is, is it the case that any E^{ce} enumerable in indices is the orbit relation of an action computable in the indices? The answer is (again): no.

Theorem (Coskey, Hamkins, R. Miller)

E_0^{ce} is enumerable in indices but there is no group action G computable in the indices so that $E_0^{ce} = E_G^{ce}$.

One way to see this is by using the following lemma. Say that a given E_G^{ce} is permutation induced if there is a computable subgroup H of S_∞ so that

$$x \, E_G^{ce} \, y \Leftrightarrow (\exists \pi \in H)(W_y = \{\pi(n) : n \in W_x\}).$$

Realizing E_0^{ce} via a group action, II

Lemma (Andrews, S.)

Every orbit relation of a group action computable in indices is permutation induced.

So, when dealing with E_G^{ce}, we shall assume that G is a subgroup of S_∞ whose action on the c.e. sets is given, for all $\pi \in G$, by

$$\pi \cdot W_x = \{\pi(n) : n \in W_x\}.$$

From the lemma, it immediately follows that no E_G^{ce} glues c.e. sets of different size. So, e.g., neither E_0^{ce} nor E_1^{ce} can be realized by group actions computable in indices. To overcome this limitation, it is natural to relax the notion of realizability and reasoning up to $\leq_c$. Then, the next question arises:

Is there G so that $E_0^{ce} \sim_c E_G^{ce}$?

Realizing E_0^{ce} via a group action, III

Since E_0^{ce} is Σ_3^0 universal, then all E_G^{ce} reduce to it. So, the question is really whether E_0^{ce} can be encoded into some E_G^{ce}.

Let P be the subgroup of S_∞ generated by all permutations with finite support (i.e., those that move only finitely many elements).

Theorem (Andrews, S.)

$E_0^{ce} \sim_c E_P$.

The proof is a priority construction dealing with Σ_3^0 approximations. Yet, note that E_P is, in a sense, the closest you may get to E_0^{ce} by using permutations. Indeed, $i \, E_P \, j$ if and only if there is n so that:

- $|W_i \cap [0,n]| = |W_j \cap [0,n]|$,
- and $W_i \smallsetminus [0,n] = W_j \smallsetminus [0,n]$.

At this point, one may suspect that "few" orbit relations would be of the highest complexity (i.e., that of E_0^{ce}). This is not the case.

In fact, we have obtained the following neat – and quite unexpected – dichotomy:

Theorem (Andrews, S.)

For all groups G acting computably in indices,

- *If G has finitely many actions, then* $E_G^{ce} \sim_c =^{ce}$,
- *If G has infinitely many actions, then* $E_G^{ce} \sim_c E_0^{ce}$.

Hence, E_∞^{ce} has many natural realizations.

Failures of Feldman-Moore and Glimm-Effros

Anyway, the analog of **Feldman-Moore** theorem fails also working up to $\leq_c$, e.g., E_{min} and E_{max} are enumerable indices but, being stricly below $=^{ce}$, they cannot be equivalent to any E_G^{ce}. In fact,

Theorem (Andrews, S.)

1. *There is an infinite chain of equivalence relations which are enumerable in the indices between* $=_{ce}$ *and* E_0^{ce}.
2. *There is an infinite antichain of equivalence relations enumerable in indices between* $=^{ce}$ *and* E_0.

Thus, there is no computable analog of **Glimm-Effros** dichotomy.

Isomorphism relations

Isomorphism relations, I

To introduce the next topic, let's briefly go back to the Borel theory.

If one considers Borel actions of *uncountable* groups, many more orbit relations arise. A notable example is given by isomorphism relations.

- For a countable language L, let $\mathrm{Mod}(L)$ denote the collection of all countable L-models with universe ω. Each element of $\mathrm{Mod}(L)$ can be viewed as an element of the product space
$$X_L := \prod_{i \in I} 2^{\omega^{n_i}},$$
which is homeomorphic to the Cantor space.

Isomorphism relations, II

- The logic action of S_∞ on X_L is given as follows:

$$\pi \cdot M \models R(x_0, \ldots, x_i)$$

if and only if

$$M \models R(\pi^{-1}(x_0), \ldots, \pi^{-1}(x_i)).$$

This action is continuous and the resulting orbit relation is just the isomorphism relation on $\mathrm{Mod}(L)$, denoted $\cong_L$.

- If an L-formula φ is $\mathcal{L}_{\omega_1\omega}$, then $\mathrm{Mod}(\varphi) \subseteq \mathrm{Mod}(L)$ is standard Borel. Then, the logic action on $\mathrm{Mod}(\varphi)$ generates $\cong_\varphi$. This allows to use Borel technology to assess the complexity of natural classes of countable structures.

The most complex classes of countable structures, I

Say that a class $\mathbb{K}$ of countable structures is on top for $\leq_B$ if, for all countable languages L, $\cong_L$ Borel reduces to $\cong_{\mathbb{K}}$. (This is the same of asking that every S_∞-relation reduces to $\cong_{\mathbb{K}}$).

It turns out that many familiar classes are on top, including:

- *Undirected graphs, trees, linear orders, nilpotent groups, fields*;
  H. Friedman, **Stanley** (1989)
- *Boolean algebras*;
  Camerlo, **Gao** (2001)
- *Torsion-free abelian groups.*
  Paolini, **Shelah** (preprint)

The most complex classes of countable structures, II

H. Friedman and Stanley named the property of being on top "Borel completeness". But this may be misleading:

1. Classes on top are not Borel (but analytic);
2. There are Borel equivalence relations which don't admit a classification by countable structures, e.g., Kechris and Louveau showed that E_1 is not Borel reducible to the isomorphism of countable graphs.

Isomorphism relations under computable lenses, I

Computable reductions are well-suited for assessing the complexity of the isomorphism problem between computable structures.

Recall that structure with universe ω is computable if its relations and functions are computable, thus such structures can be identified with a natural number. For a class $\mathbb{K}$, $I(\mathbb{K}) \subseteq \omega$ denotes the collection of indices of computable structures from $\mathbb{K}$.

Then, to compare isomorphism relations on computable structures, one considers partial computable reductions with domain containing the relevant set $I(\mathbb{K})$.

Isomorphism relations under computable lenses, II

Theorem (Fokina, Sy Friedman, Harizanov, Knight, McCoy, Montalbán)

Isomorphism relations on the following classes of computable structures are Σ^1_1 universal:

- *Trees, undirected graphs, nilpotent groups, fields (as for $\leq_B$);*
- *But also torsion-free abelian groups and torsion abelian groups.*

Hence, this contrasts with the Borel theory in two ways:

1. Every hyperarimetic relation (on ω) admits a classification by computable structures;
2. Torsion abelian groups are *not* on top for $\leq_B$, but computable torsion abelian groups are on top for $\leq_c$.

The Friedman-Stanley jump

To gauge the complexity of Borel isomorphism relations, **H. Friedman** and **Stanley** introduced the following jump operator:

- Let E be on a standard Borel space X. The FS-jump of E, denoted E^+, is the equivalence relation on X^ω given by

$$(x_n)\, E^+ (y_n) \Leftrightarrow \{[x_n]_E : n \in \omega\} = \{[y_n]_E : n \in \omega\}.$$

This jump is proper on Borel equivalence relations:

Theorem (H. Friedman, Stanley)

If E is a Borel and it has more than one class, then $E <_B E^+$.

The computable FS-jump

Clemens, Coskey, and Krakoff (2022) introduced a natural computable analog of the FS-jump:

- For E on ω, E^+ is given by

$$xE^+y \Leftrightarrow [W_x]_E = [W_y]_E.$$

That is, intuitively W_x and W_y are E^+-equivalent if they list the same E-classes. Note that $\mathrm{Id}(\omega)^+ \sim_c =^{ce}$.

Theorem (Clemens, Coskey, Krakoff)

- *If E is universal Σ^1_1, then $E \sim_c E^+$.*
- *If E is hyperarithmetic, then $E <_c E^+$.*

On the computable *FS*-tower, I

The Friedman-Stanley tower is obtained by starting with the identity on a standard Borel space and then iterating the *FS*-jump transfinitely, along the countable ordinals.
H. Friedman and **Stanley** proved that the *FS*-tower is cofinal for all Borel isomorphism relations.

In the computable setting, the transfinite jump hierarchy is similarly defined along the computable ordinals.

Formally, for $a \in \mathcal{O}$ and E an equivalence relation, E^{+a} is defined by induction as follows:

- If $a = 2^b$ then $E^{+a} = (E^{+b})^+$;
- If $a = 3 \cdot 5^e$, then $E^{+a} = \bigoplus_i E^{+\varphi_e(i)}$.

On the computable *FS*-tower, II

Theorem (Andrews, S.)

Let E on ω be hyperarithmetic. Then, there exists a notation $a \in \mathcal{O}$ such that

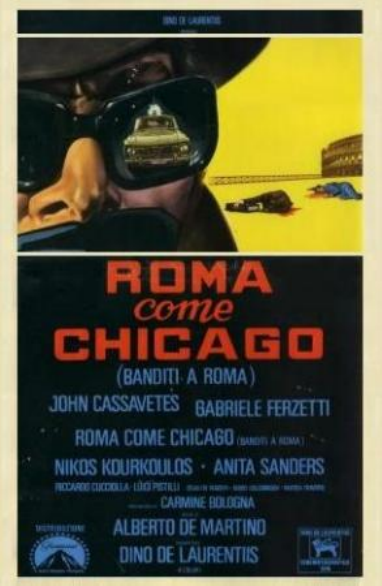
$$E \leq_c \mathsf{Id}(\omega)^{+a}.$$

Let's close this topic by mentioning that the computable *FS*-jump is notation dependant:

Theorem (Andrews, S.)

- *Let $a, b \in \mathcal{O}$ be notations for $\alpha < \omega^2$. Then, $E^{+a} \sim_c E^{+b}$ for all E.*
- *There are two notations $a, b \in \mathcal{O}$ for ω^2 so that $\mathsf{Id}(\omega)^{+a}$ and $\mathsf{Id}(\omega)^{+b}$ are incomparable.*

Thank you!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