The first-order part of computational problems

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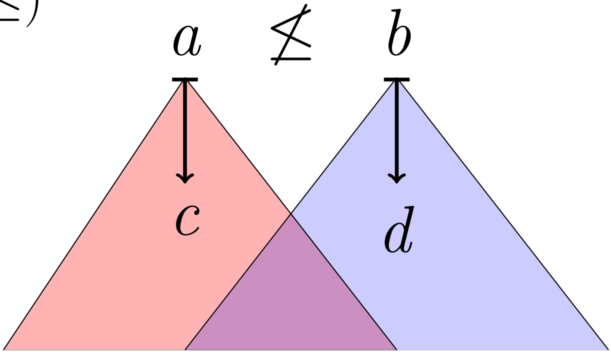
Motivation

$(P, \leq)$



If c and d are maxima (in the resp. lower cones) satisfying some property φ then

$$c \not\leq d \Rightarrow a \not\leq b$$

Weihrauch reducibility

Computational problem: partial multi-valued function $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$

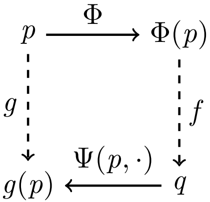
input : any $x \in \operatorname{dom}(f)$

output : any $y \in f(x)$

More general spaces can be considered, but problems on $\mathbb{N}^{\mathbb{N}}$ are enough to study Weihrauch degrees.

$g \leq_{\mathrm{W}} f :\iff$ there are computable $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ s.t.

- Given $p \in \operatorname{dom}(g)$, $\Phi(p) \in \operatorname{dom}(f)$
- Given $q \in f(\Phi(p))$, $\Psi(p, q) \in g(p)$



$g \leq_{\mathrm{sW}} f :\iff g \leq_{\mathrm{W}} f$ and Ψ does not depend on p.

First-order problems

A computational problem $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}$ can be identified with the problem $\mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$

$$\underset{\text{dom}(f)}{\overset{p}{\cap}} \mapsto \{q \in \mathbb{N}^\mathbb{N} \, : \, q(0) \in f(p)\}$$

If g has codomain Y and there is a computable injection $Y \to \mathbb{N}$ with computable inverse we say that it is *first-order*.

The first-order part of a problem

Theorem (Dzhafarov, Solomon, Yokoyama)

For every f, $\max_{\leq_W}\{g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N} \,:\, g \leq_W f\}$ *is well-defined.*

Proof.

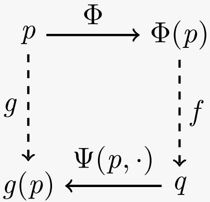
Assume $g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ is s.t. $g \leq_W f$ via Φ, Ψ.

Φ maps $p \in \mathrm{dom}(g)$ to an input for f.

Given p, we can uniformly compute an index $w \in \mathbb{N}^{\mathbb{N}}$ for the map $q \mapsto \Psi(p, q)$.

$g(p) \subset \mathbb{N}$, hence for every $q \in f(\Phi(p))$,

$$\Phi_w(q)(0) = \Psi(p, q)(0) \downarrow \in g(p)$$



The first-order part of a problem

Theorem (Dzhafarov, Solomon, Yokoyama)

For every f, $\max_{\leq_W}\{g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N} : g \leq_W f\}$ *is well-defined.*

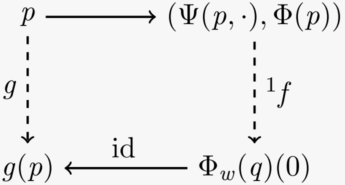
Proof.

Define ${}^1f :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}$ as

$$ {}^1f(w, x) := \{\Phi_w(q)(0) : q \in f(x)\}. $$

Intuitively: ${}^1f$ behaves just like f but stops at the first digit!

It follows that $g \leq_W {}^1f \leq_W f$.



□

The first-order part of a problem

For $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$, we define ${}^1f :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ as:

input : (w, x) s.t. $x \in \mathrm{dom}(f)$ and, for every solution $q \in f(x)$, $\Phi_w(q)(0)\downarrow$

output : any n s.t. $\Phi_w(q)(0) = n$ for some $q \in f(x)$

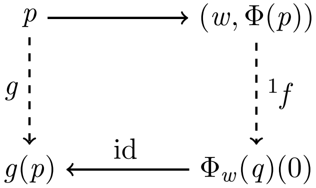
${}^1(\cdot)$ is an interior operator:

- ${}^1({}^1f) \equiv_{\mathrm{W}} {}^1f \leq_{\mathrm{W}} f$
- $f \leq_{\mathrm{W}} g \Rightarrow {}^1f \leq_{\mathrm{W}} {}^1g$

In particular, ${}^1f \not\equiv_{\mathrm{W}} {}^1g \Rightarrow f \not\equiv_{\mathrm{W}} g$.

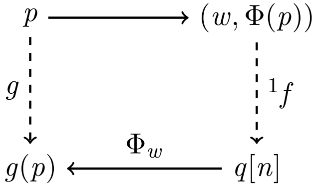
Computing prefixes

${}^1 f$ computes "sufficiently long" prefixes of solutions.



By the continuity of $\Phi_w = \Psi(p, \cdot)$, only a prefix of q is needed to solve g.

$q[n]$ is sufficiently long so that Φ_w converges on 0.



A few benchmarks

Choice problems are pivotal in the Weihrauch lattice.

Given $A \neq \emptyset$ (with some properties), find $x \in A$.

- C_k: given a sequence in $(k+1)^{\mathbb{N}}$, find a number that is not enumerated.
- $\mathsf{C}_{\mathbb{N}}$: same as C_k but with no bound.
- $\mathsf{C}_{2^{\mathbb{N}}}$: given a tree $T \subset 2^{<\mathbb{N}}$, find a path $p \in [T]$ (WKL).
- $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$: given a tree $T \subset \mathbb{N}^{<\mathbb{N}}$, find a path $p \in [T]$.
- $\boldsymbol{\Sigma}^1_1$-$\mathsf{C}_{\mathbb{N}}$: given a list of subtrees of $\mathbb{N}^{<\mathbb{N}}$, find the index of an ill-founded one.

Operations on problems

$f \times g$	:	solve f and g in parallel
f^*	:	solve finitely many instances of f in parallel
$\widehat{f}$	:	solve infinitely many instances of f in parallel
$f * g$	:	solve g, apply some computable functional, then solve f
f'	:	*jump in the Weihrauch lattice*
		name of input: a sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^\mathbb{N}$ s.t. $\lim_n p_n$ is a name for an instance x of f; output : $f(x)$
$\vdots$		

Examples

- (Brattka, Pauly) if f is densely realized (for every p, $f(p)$ is dense) then ${}^1 f \le_W \mathrm{id}$. Examples: "given p, produce q which is non-computable/non-hyp/ML-random relative to p".

- (Brattka, Gherardi, Marcone) ${}^1\mathsf{lim} \equiv_W \mathsf{C}_{\mathbb{N}}$:

  Given $(p_n)_{n\in\mathbb{N}}$, for each i we can compute $\lim_n p_n(i)$ with finitely many mind changes.

  Being $\le_W \mathsf{C}_{\mathbb{N}}$ corresponds to being uniformly computable with finitely many mind changes, hence ${}^1\mathsf{lim} \le_W \mathsf{C}_{\mathbb{N}}$. The other reduction follows from $\mathsf{lim} \equiv_W \widehat{\mathsf{C}_{\mathbb{N}}}$.

Examples

- (Dzhafarov, Solomon, Yokoyama) ${}^1\mathsf{WWKL} \equiv_{\mathrm{W}} {}^1\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_2^*$: exploits the compactness of $2^{\mathbb{N}}$.

- (Goh, Pauly, V.) ${}^1\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathbf{\Sigma}_1^1\text{-}\mathsf{C}_{\mathbb{N}}$:

  Given a tree $T \subset \mathbb{N}^{<\mathbb{N}}$, we look for a sufficiently long σ that extends to a path in T ($\Sigma_1^{1,T}$ condition).

How about $\mathbf{\Pi}_1^1\text{-}\mathsf{CA} \equiv_{\mathrm{W}} \widehat{\mathsf{WF}}$?

Is there a general rule?

FOP and parallelization

Assume $f = \widehat{g}$ for some first-order g.

Can we characterize ${}^1f$ in terms of g?

$$\mathsf{lim} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_{\mathbb{N}}} \longrightarrow {}^1\mathsf{lim} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}^*$$

$$\mathsf{WKL} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_2} \longrightarrow {}^1\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_2^*$$

Guess: ${}^1f \equiv_{\mathrm{W}} g^*$

This doesn't quite work: consider

"Given a sequence of trees in $\mathbb{N}^{<\mathbb{N}}$, return 0 if they are all well-founded, or return $i+1$ s.t. the i-th tree is ill-founded"

$\mathbf{\Pi}^1_1$-CA can solve it, but WF^* cannot.

The unbounded-* operator

We define the *unbounded finite parallelization*. Intuitively: just like $(\cdot)^*$, but with no commitment to the number of instances.

For $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$

$f^{u*} :\subseteq \mathbb{N}^\mathbb{N} \times (\mathbb{N}^\mathbb{N})^\mathbb{N} \rightrightarrows (\mathbb{N}^\mathbb{N})^{<\mathbb{N}}$ is the following problem:

$$(w, (x_n)_{n\in\mathbb{N}}) \mapsto \{(y_n)_{n<k} : (\forall n < k)(y_n \in f(x_n)) \text{ and } \Phi_w(\langle y_n \rangle_{n<k})(0)\downarrow\}$$

$(\cdot)^{u*}$ is a closure operator:

- $f \le_{\mathrm{W}} f^{u*} \equiv_{\mathrm{W}} (f^{u*})^{u*}$
- $f \le_{\mathrm{W}} g \Rightarrow f^{u*} \le_{\mathrm{W}} g^{u*}$

Moreover $f^* \le_{\mathrm{W}} f^{u*} \le_{\mathrm{W}} \widehat{f}$

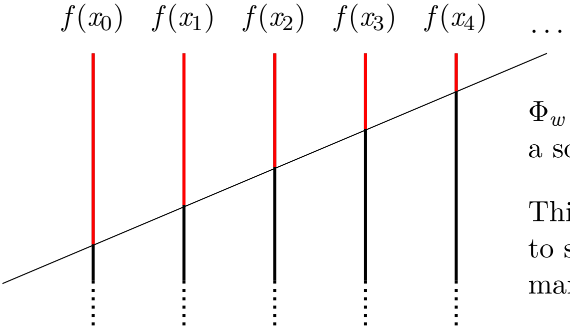
FOP and parallelization

Theorem (Soldà, V.)

For every f, ${}^1(\widehat{f}) \equiv_{\mathrm{W}} {}^1(f^{u*})$.

Proof (Sketch): ${}^1(f^{u*}) \le_{\mathrm{W}} {}^1(\widehat{f})$ is easy as $f^{u*} \le_{\mathrm{W}} \widehat{f}$.

${}^1(\widehat{f}) \le_{\mathrm{W}} {}^1(f^{u*})$: let $(w, (x_n)_n)$ be an input for ${}^1(\widehat{f})$.



Φ_w selects a prefix of a solution.

This corresponds to selecting finitely many columns.

FOP and parallelization

Theorem (Soldà, V.)

For every f, if $f \equiv_{\mathrm{W}} \widehat{g}$ for some first-order g, then

$$^1 f \equiv_{\mathrm{W}} {}^1(g^{u*}) \equiv_{\mathrm{W}} ({}^1 g)^{u*} \equiv_{\mathrm{W}} g^{u*}$$

If $\mathrm{id} \le_{\mathrm{sW}} f$ *then this lifts to jumps: for every n*

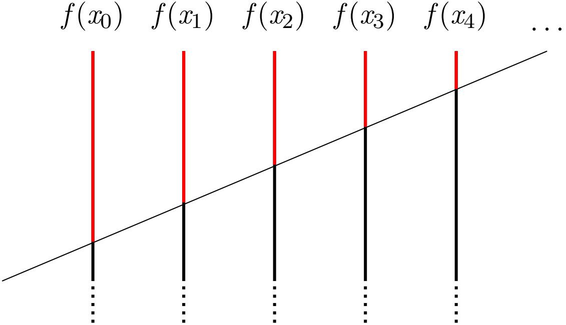
$$^1(f^{(n)}) \equiv_{\mathrm{sW}} (g^{u*})^{(n)}$$

In other words: ${}^1(\cdot)$ and $(\cdot)^{u*}$ commute for first-order problems.

Is this peculiar of first-order problems?

FOP and unbounded-*

Remark: let $(w, (x_n)_n)$ be an input for ${}^1(\widehat{f})$.



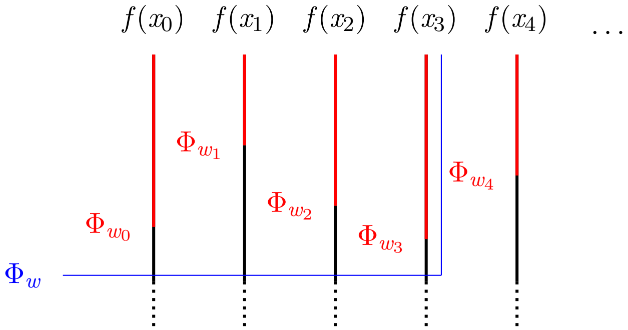
Φ_w selects a prefix of a solution.

The prefix of $f(x_i)$ may depend on the solution to x_j.

FOP and unbounded-*

On the other hand, an input for $({}^1 f)^{u*}$ is $(w, (w_n, x_n)_n)$ s.t.

- for every n, Φ_{w_n} selects a prefix of $f(x_n)$
- Φ_w selects "finitely many prefixes"



The prefix of $f(x_i)$ is independent of the solution of x_j.

FOP and unbounded-*

In some cases, we have a work around. E.g. if $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is finitely valued (for every $p \in \mathrm{dom}(f)$, $|f(p)| < \infty$) then

$$^1(\widehat{f}) \equiv_{\mathrm{W}} (^1f)^{u*}.$$

Proposition (Soldà, V.)

There is f s.t. $^1(\widehat{f}) \not\leq_{\mathrm{W}} (^1f)^{u*}$.

Lemma

There are two sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ s.t.

- *for every n, $\emptyset' \not\leq_T A_n$, $\emptyset' \not\leq_T B_n$, but $\emptyset' \leq_T A_n \oplus B_n$;*
- *for every n and every computable functional Ψ s.t. $\emptyset' = \Psi(\langle A_i \rangle, B_n)$, the map sending x to the prefix of B_n used in the computation of $\emptyset'(x)$ is not B_n-computable.*

Applications

- $\mathsf{lim} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_{\mathbb{N}}} \equiv_{\mathrm{W}} \widehat{\mathsf{LPO}}$. Since $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}^{u*}$ (Neumann, Pauly)

$$^{1}(\mathsf{lim}) \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{LPO}^{u*}$$

  This lifts to jumps: for every n

$$^{1}(\mathsf{lim}^{(n)}) \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}^{(n)} \equiv_{\mathrm{W}} (\mathsf{LPO}^{(n)})^{u*}$$

- $\mathbf{\Pi}^1_1\text{-}\mathsf{CA} \equiv_{\mathrm{W}} \widehat{\mathsf{WF}}$, hence $^{1}\mathbf{\Pi}^1_1\text{-}\mathsf{CA} \equiv_{\mathrm{W}} \mathsf{WF}^{u*}$.

The diamond operator

(Neumann, Pauly) $f^\diamond$ (roughly): "computation using f as oracle"

(Hirschfeldt, Jockusch): define the game $G(f \to g)$

Player 1	Player 2
$p \in \mathrm{dom}(g)$	
	p-computable $q \in g(p)$ and wins OR
	p-computable $x_1 \in \mathrm{dom}(f)$
$y_1 \in f(x_1)$	
	$\langle p, y_1 \rangle$-computable $q \in g(p)$ and wins OR
	$\langle p, y_1 \rangle$-computable $x_2 \in \mathrm{dom}(f)$
	$\vdots$

Player 2 wins if he declares victory. Otherwise Player 1 wins.

$g \leq_{\mathrm{W}} f^\diamond$ iff Player 2 has a computable winning strategy for $G(f \to g)$

unbounded-* and diamond

The diamond is essentially an "unbounded compositional product".

What is the relation between f^{u*} and $f^{\diamond}$?

Observation: for every f we have $f^{u*} \le_{\mathrm{W}} f^{\diamond}$

Can we have $f^{u*} \equiv_{\mathrm{W}} f^{\diamond}$?

Theorem (Soldà, V.)

If $f\colon \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ *is s.t.* $\{(x,n) : n \in f(x)\} \in \Pi^0_1$ *then* $f^{u*} \equiv_{\mathrm{W}} f^{\diamond}$.
Besides, if $\mathrm{ran}(f) = k$ *then* $f^{*} \equiv_{\mathrm{W}} f^{\diamond}$.

Idea: we guess the possible answers to the oracle calls and use the effective closedness of $\mathrm{Graph}(f)$ to discard wrong guesses.

Examples: C_k for every $k \in \mathbb{N}$.

unbounded-* and diamond

(Brattka, Gherardi) The *completion* of a represented space X is

$$\overline{X} := X \cup \{\bot\}$$

Intuitively, we have the possibility to postponing information about $x \in X$. Doing so indefinitely results in a name of $\bot$.

Using the completion $\overline{f} \colon \overline{X} \rightrightarrows \overline{Y}$ of f :

Theorem (Soldà, V.)

For every complete $f :\subseteq X \rightrightarrows \mathbb{N}$, $f^{u*} \equiv_{\mathrm{W}} f^{\diamond}$

Examples: LPO, WF.

Question: can we do better?

Applications

- $\mathsf{WKL} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_2}$:

$$^1(\mathsf{WKL}) \equiv_{\mathrm{W}} (\mathsf{C}_2)^{u*} \equiv_{\mathrm{W}} \mathsf{C}_2^*$$

- This lifts to jumps: $\mathsf{WKL}^{(n)} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_2^{(n)}}$

$$^1(\mathsf{WKL}^{(n)}) \equiv_{\mathrm{W}} (\mathsf{C}_2^{(n)})^{u*} \equiv_{\mathrm{W}} (\mathsf{C}_2^*)^{(n)}$$

- $\mathbf{\Pi}^1_1\text{-}\mathsf{CA} \equiv_{\mathrm{W}} \widehat{\mathsf{WF}}$:

$$\mathsf{WF}^* <_{\mathrm{W}} \mathsf{WF}^{u*} \equiv_{\mathrm{W}} \mathsf{WF}^{\diamond} \equiv_{\mathrm{W}} {}^1\mathbf{\Pi}^1_1\text{-}\mathsf{CA}$$

Ramsey's theorem

Theorem (Brattka, Rakotoniaina)

For every $n > 1$ and $k \geq 2$, $\mathsf{C}_k^{(n)} \leq_{\mathrm{W}} \widehat{\mathsf{SRT}_k^n} \leq_{\mathrm{W}} \widehat{\mathsf{RT}_k^n} \equiv_{\mathrm{W}} \mathsf{WKL}^{(n)}$

Corollary (Soldà, V.)

For every $n > 1$ and $k \geq 2$, $\mathsf{C}_k^{(n)} <_{\mathrm{W}} {}^1\mathsf{SRT}_k^n \leq_{\mathrm{W}} {}^1\mathsf{RT}_k^n \leq_{\mathrm{W}} (\mathsf{C}_2^*)^{(n)}$

The first reduction is strict as witnessed by $\mathsf{C}_{\mathbb{N}}$.

Are the last two reductions strict?

Ramsey's theorem

Open question (Brattka, Rakotoniaina): $\mathsf{C}_{\mathbb{N}}' \le_W \mathsf{RT}^2_2$?

Theorem (Soldà, V.)

For every n and $k > 1$, $\mathsf{C}_{\mathbb{N}}^{(n)} \not\le_W \mathsf{RT}^{n+1}_k$.

In particular, for $n = 2$, ${}^1\mathsf{SRT}^2_2 <_W {}^1\mathsf{RT}^2_2$ (Brattka, Rakotoniaina), hence

Theorem (Soldà, V.)

$\mathsf{C}_2'' <_W {}^1\mathsf{SRT}^2_2 <_W {}^1\mathsf{RT}^2_2 <_W (\mathsf{C}_2^*)'' \equiv_W {}^1(\mathsf{WKL}'')$

Can we fully characterize ${}^1\mathsf{RT}^n_k$?

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