Infinite-dimensional structural Ramsey theory

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A question of Kechris–Pestov–Todorcevic

Develop infinite-dimensional Ramsey theory for Fraïssé structures.

Ramsey's Theorem (finite-dimensional)

Theorem (Ramsey)

Given m, r and a coloring of $[\omega]^m$ into r colors, there is an $N \in [\omega]^\omega$ such that all members of $[N]^m$ have the same color.

$$\forall m \, \forall r, \quad \omega \rightarrow (\omega)^m_r$$

This is finite-dimensional because the objects being colored are finite.

Infinite-dimensional Ramsey Theory

A subset $\mathcal{X}$ of $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that

$$[N]^\omega \subseteq \mathcal{X} \ \text{ or } \ [N]^\omega \cap \mathcal{X} = \emptyset.$$

AC $\Rightarrow \exists \mathcal{X} \subseteq [\omega]^\omega$ which is not Ramsey.

Solution: restrict to 'definable' sets.

Infinite-dimensional Ramsey Theory

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$$[N]^\omega \subseteq \mathcal{X} \text{ or } [N]^\omega \cap \mathcal{X} = \emptyset.$$

Nash-Williams Thm. Clopen sets are Ramsey.

Galvin–Prikry Thm. Borel sets are Ramsey.

Silver Thm. Analytic sets are Ramsey.

Ellentuck Thm. A set if Ramsey iff it has the property of Baire in the Ellentuck topology.

$$\omega \to_* (\omega)^\omega$$

Ellentuck Theorem

The **Ellentuck topology** is generated by basic open sets of the form

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

Ellentuck Thm. A set $\mathcal{X} \subseteq [\omega]^\omega$ satisfies

$(*) \qquad \forall [s, A] \ \exists B \in [s, A]$ such that $[s, B] \subseteq \mathcal{X}$ or $[s, B] \cap \mathcal{X} = \emptyset$

iff $\mathcal{X}$ has the property of Baire with respect to the Ellentuck topology.

$(*)$ is called completely Ramsey by Galvin–Prikry and Ramsey by Todorcevic.

The Ellentuck space is the prototype for **topological Ramsey spaces**: Points are infinite sequences, topology is induced by finite heads and infinite tails, and every subset with the property of Baire satisfies $(*)$.

Homogeneous and Universal Structures

A structure **K** is **homogeneous** if every isomorphism between two finite induced substructures of **K** extends to an automorphism of **K**.

A structure **K** is **universal** for a class of structures $\mathcal{K}$ if every structure in $\mathcal{K}$ embeds into **K**.

Homogeneous universal structures are Fraïssé limits and include

- $(\mathbb{Q}, <)$ The rationals
- $(\mathcal{R}, E)$ The Rado graph
- $(\mathcal{H}_3, E)$ The triangle-free Henson graph

Homogeneous structures are good environments for Ramsey theory.

KPT Question revisited

Problem 11.2 in (KPT 2005). Given a homogeneous structure $\mathbf{K}$ and some natural topology on $\binom{\mathbf{K}}{\mathbf{K}}$, find the right notion of 'definable set' so that all definable sets are Ramsey.

$$\mathbf{K} \rightarrow_* (\mathbf{K})^{\mathbf{K}}$$

Theorem

Theorem (D.)

Let **K** *be a Fraïssé structure with universe* $\mathbb{N}$ *satsfying* SDAP^+ *with finitely many relations of arity at most two.*

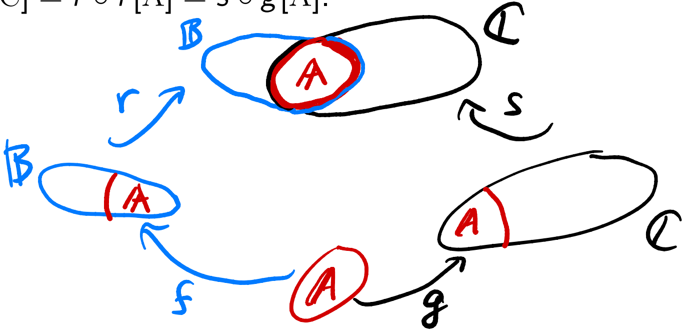
- *There are natural (and seemingly necessary) subspaces of* $\binom{\mathbf{K}}{\mathbf{K}}$ *in which every Borel set is completely Ramsey.*
- *As a corollary we recover exact big Ramsey degrees.*
- *Under an additional rigidity-like property, an analogue of the Ellentuck theorem holds.*

Examples. Rado graph, generic k-partite graphs, ordered versions.

Ellentuck analogues hold for $\mathbb{Q}$, $\mathbb{Q}_n$, $\mathbb{Q}_{\mathbb{Q}}$.

Disjoint and Free Amalgamation Properties

A Fraïssé class $\mathcal{K}$ satisfies the **Disjoint Amalgamation Property** if, given embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, there is an amalgam $\mathbf{D} \in \mathcal{K}$ with embeddings $r : \mathbb{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ such that $r \circ f = s \circ g$ and moreover, $r[\mathrm{B}] \cap s[\mathrm{C}] = r \circ f[\mathrm{A}] = s \circ g[\mathrm{A}]$.



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A Fraïssé class satisfies the **Free Amalgamation Property** if it satisfies DAP and **D** can be taken to have no additional relations on its universe other than those inherited from **B** and **C**.

Substructure Free Amalgamation Property

A Fraïssé class $\mathcal{K}$ satisfies **SFAP** if $\mathcal{K}$ has free amalgamation, and given $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{K}$, the following holds: Suppose

(1) $\mathbf{A}$ is a substructure of $\mathbf{C}$, where $\mathbf{C}$ extends $\mathbf{A}$ by two vertices, say $\mathrm{C} \setminus \mathrm{A} = \{v, w\}$;

(2) $\mathbf{A}$ is a substructure of $\mathbf{B}$ and σ and τ are 1-types over $\mathbf{B}$ with $\sigma \restriction \mathbf{A} = \mathrm{tp}(v/\mathbf{A})$ and $\tau \restriction \mathbf{A} = \mathrm{tp}(w/\mathbf{A})$; and

(3) $\mathbf{B}$ is a substructure of $\mathbf{D}$ which extends $\mathbf{B}$ by one vertex, say v', such that $\mathrm{tp}(v'/\mathbf{B}) = \sigma$.

Then there is an $\mathbf{E} \in \mathcal{K}$ extending $\mathbf{D}$ by one vertex, say w', such that $\mathrm{tp}(w'/\mathbf{B}) = \tau$, $\mathbf{E} \restriction (\mathrm{A} \cup \{v', w'\}) \cong \mathbf{C}$, and $\mathbf{E}$ adds no other relations over D.

Substructure Disjoint Amalgamation Property

A Fraïssé class $\mathcal{K}$ has **SDAP** if $\mathcal{K}$ has DAP and

whenever **C** extends **A** by two vertices, v, w, there exist $\mathbf{A}', \mathbf{C}'$ such that $\mathbf{A} \leq \mathbf{A}'$ and $\mathbf{C}'$ is a disjoint amalgamation of $\mathbf{A}'$ and **C** over **A** such that, letting $\{v', w'\} = \mathrm{C}' \setminus \mathrm{A}'$, if

(1) **B** is any structure containing $\mathbf{A}'$ as a substructure, and σ and τ are 1-types over **B** satisfying $\sigma \restriction \mathbf{A}' = \mathrm{tp}(v'/\mathbf{A}')$ and $\tau \restriction \mathbf{A}' = \mathrm{tp}(w'/\mathbf{A}')$;

(2) **D** extends **B** by one vertex, v'', such that $\mathrm{tp}(v''/\mathbf{B}) = \sigma$;

then there is an **E** extending **D** by one vertex, w'', such that $\mathrm{tp}(w''/\mathbf{B}) = \tau$ and $\mathbf{E} \restriction (\mathrm{A} \cup \{v'', w''\}) \cong \mathbf{C}$.

SDAP$^+$ and LSDAP$^+$

What are these subspaces of $\binom{\mathbf{K}}{\mathbf{K}}$?

Big Ramsey degrees pose constraints.

We'll start with the Ramsey Property for finite structures.

Finite Structural Ramsey Property

For structures $\mathbf{A}, \mathbf{B}$, write $\mathbf{A} \leq \mathbf{B}$ iff $\mathbf{A}$ embeds into $\mathbf{B}$.

$\binom{\mathbf{B}}{\mathbf{A}}$ denotes the set of all copies of $\mathbf{A}$ in $\mathbf{B}$.

A class $\mathcal{K}$ of finite structures has the **Ramsey Property** if given $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ with $\mathbf{A} \leq \mathbf{B}$, and $2 \leq m$, there is a large enough $\mathbf{C} \in \mathcal{K}$ so that for any coloring of $\binom{\mathbf{C}}{\mathbf{A}}$ into m colors, there is some $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ so that all copies of $\mathbf{A}$ in $\mathbf{B}'$ have the same color.

Lots of work done! (seminal result of Nešetřil–Rödl, 1977/83)

Examples: The classes of finite linear orders, finite ordered graphs, and finite ordered k-clique-free graphs have RP.

Finite-Dimensional Structural Ramsey Theory

Let $\mathbf{K}$ be an infinite homogeneous structure.

$\mathbf{K}$ has **finite big Ramsey degrees** if for each finite substructure $\mathbf{A}$ of $\mathbf{K}$, there is an integer $T \geq 1$ so that for any coloring of $\binom{\mathbf{K}}{\mathbf{A}}$ into finitely many colors, there is a subcopy $\mathbf{K}'$ of $\mathbf{K}$ such that $\binom{\mathbf{K}'}{\mathbf{A}}$ takes no more than T colors.

The **big Ramsey degree** of $\mathbf{A}$ in $\mathbf{K}$, denoted by $T(\mathbf{A}, \mathbf{K})$, is the least such positive integer T.

- If $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$ s.t. $|\mathrm{Aut}(\mathbf{K})| > 1$, then $\exists \mathbf{A} \in \mathcal{K}$ with $T(\mathbf{A}, \mathbf{K}) > 1$ (or infinite). (Hjorth 2008)

Previous Big Ramsey Degree results

- 1933. $T(\text{Pairs}, \mathbb{Q}) \geq 2$. (Sierpiński)
- 1975. $T(\text{Edge}, \mathcal{R}) \geq 2$. (Erdős, Hajnal, Pósa)
- 1979. $(\mathbb{Q}, <)$: All BRD computed. (D. Devlin)
- 1986. $T(\text{Vertex}, \mathcal{H}_3) = 1$. (Komjáth, Rödl)
- 1989. $T(\text{Vertex}, \mathcal{H}_n) = 1$. (El-Zahar, Sauer)
- 1996. $T(\text{Edge}, \mathcal{R}) = 2$. (Pouzet, Sauer)
- 1998. $T(\text{Edge}, \mathcal{H}_3) = 2$. (Sauer)
- 2006, 2008. The Rado graph: All BRD characterized; computed. (Laflamme, Sauer, Vuksanović); (J. Larson)
- 2010. Dense Local Order $\mathbf{S}(2)$: All BRD computed. (Laflamme, Nguyen Van Thé, Sauer)

Developments via coding trees and forcing

- 2017. Triangle-free Henson graphs: Upper Bounds. (D.)
- 2019. k-clique-free Henson graphs: Upper Bounds. (D.)
- 2019. Infinite-dimensional Ramsey theory for Borel sets of Rado graphs. (D.)
- 2020. SDAP$^+$ implies Exact BRD characterized by diagonal antichains. (Coulson, D., Patel)
- 2020. Binary rel. Forb($\mathcal{F}$), finite $\mathcal{F}$: Upper Bounds. (Zucker)
- 2021. Binary rel. Forb($\mathcal{F}$), finite $\mathcal{F}$: Exact BRD. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker)
- 2022. ∞-dimensional Ramsey theory of homogeneous structures with SDAP$^+$. (recovers exact BRD). (D.)
- 2022*. ∞-dimensional Ramsey theory for binary relational Flim(Forb($\mathcal{F}$)), finite $\mathcal{F}$. (recovers exact BRD). (D., Zucker)

Developments not using forcing

- 2018. Certain homogeneous metric spaces: Upper Bounds. (Mašulović) category theory.
- 2019. 3-uniform hypergraphs: Upper Bounds. (Balko, Chodounský, Hubička, Konečný, Vena) Milliken Theorem.
- 2020. Circular directed graphs: Exact BRD Computed. (Dasilva Barbosa) category theory.
- 2020. Homogeneous partial order: Upper Bounds. (Hubička) Ramsey space of parameter words. First non-forcing proof for $\mathcal{H}_3$.
- 2021. Homogenous graphs with forbidden cycles (metric spaces): Upper Bounds. (Balko, Chodounský, Hubička, Konečný, Nešetřil, Vena) parameter words.
- 2021. Homogeneous partial order: Exact BRD. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker) parameter words.
- 2022. Forb($\mathcal{F}$) binary and other arities. Upper Bounds. (BCDHKNTVZ) New methods.

Differences between what the methods produce

- Results using coding trees produce Ramsey theorems on homogeneous universal structures.
- Results which are purely combinatorial (so far) produce Ramsey theorems on universal structures which are then applied to prove upper bounds for big Ramsey degrees.
- Big questions are to
  (1) obtain purely combinatorial proofs for big Ramsey degrees;
  (2) obtain purely combinatorial proofs for Ramsey theorems on homogeneous universal structures, as these are what will lead to infinite-dimensional Ramsey theorems.

Motivations for the work presented today

(1) KPT question on ∞-dimensional Ramsey theory.

(2) My 2019 work beginning to answer a KPT question for the Rado graph, and Todorcevic's refined question to prove an ∞-dimensional Ramsey theorem which directly recovers the big Ramsey degrees.

(3) The following work with Coulson and Patel for SDAP$^+$ structures.

A Theorem of Coulson–D.–Patel

Theorem (CDP 2020)

Let **K** *be a Fraïssé structure with finitely many relations of arity at most two and satisfying* SFAP, SDAP$^+$, *or* LSDAP$^+$. *Given a finite substructure* $\mathbf{A} \leq \mathbf{K}$, *the big Ramsey degree of* **A** *equals the number of similarity types of diagonal antichains in the coding tree of* 1*-types for* **K**.

Remark. Proves a Ramsey theorem on diagonal coding trees which immediately produces exact big Ramsey degrees (no envelopes).

Examples. $\mathbb{Q}$, $\mathbb{Q}_n$, $\mathbb{Q}_{\mathbb{Q}}$, $\mathbb{Q}_{\mathbb{Q}_{\mathbb{Q}}}$, ...
Rado graph, generic k-partite graph, ordered versions, ...

Coding trees of 1-types: unavoidable

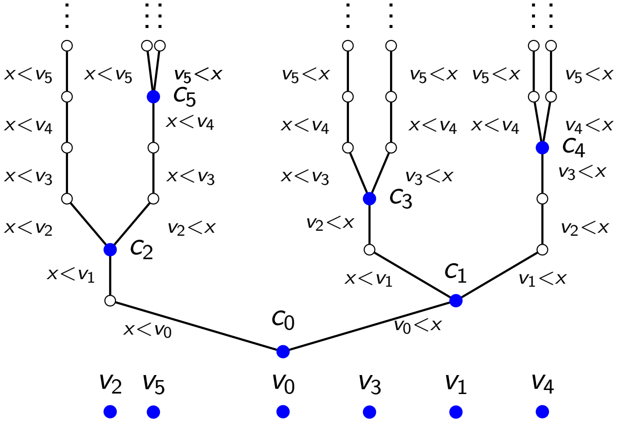
Well-ordering the vertices of $\mathbf{K}$ induces a coding tree of 1-types.

Let $\mathbf{K}$ be a homogeneous structure with finitely many relations of arity at most two and vertices $\langle v_n : n < \omega \rangle$. Let $\mathbf{K}_n$ denote $\mathbf{K} \restriction \{v_i : i < n\}$.

The **coding tree of 1-types** $\mathbb{S}(\mathbf{K})$ is the set of all complete 1-types over $\mathbf{K}_n$, $n < \omega$, along with a function $c : \omega \to \mathbb{S}(\mathbf{K})$ where $c(n)$ is the 1-type of v_n over $\mathbf{K}_n$. The tree-ordering is inclusion.

Each $s \in \mathbb{S}(\mathbf{K})$ determines a unique sequence $\langle s(i) : i < |s| \rangle$, where $s(0)$ is the 1-type over the empty structure such that $s(0) \subseteq s$, and for $1 \leq i < |s|$, $s(i)$ is the set of formulas in $s \restriction \mathbf{K}_i$ with parameter v_{i-1}.

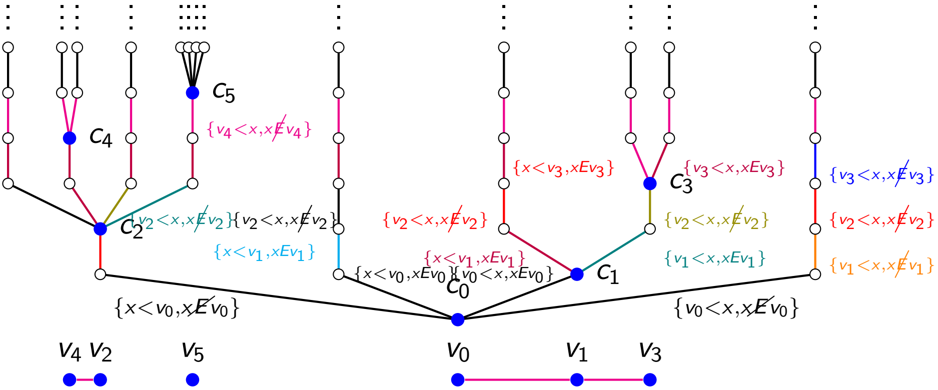
Coding Tree of 1-types for $(\mathbb{Q}, <)$



$c_0 = \emptyset$. $c_1 = \{(v_0 < x)\}$. $c_2 = \{(x < v_0), (x < v_1)\}$.
$c_3 = \{(v_0 < x), (x < v_1), (v_2 < x)\}$.

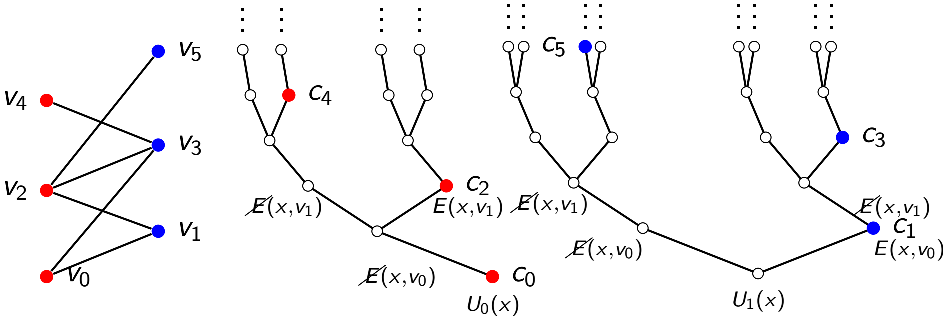
Coding tree of 1-types for $\mathbb{Q}_{\mathbb{Q}}$

Language $\mathcal{L} = \{<, E\}$. The equivalence classes are convex.



$c_0 = \emptyset$. $c_1 = \{v_0 < x, xEv_0\}$. $c_2 = \{x < v_0, x\not E v_0, x < v_1, x\not E v_1\}$.

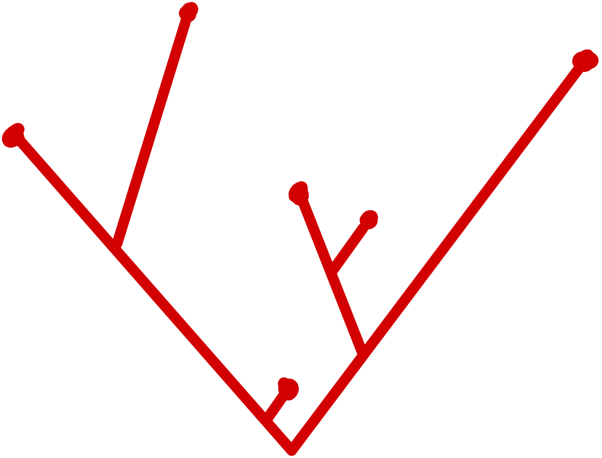
Coding tree of 1-types for the generic bipartite graph



$c_0 = \{U_0(x)\}$. $c_1 = \{U_1(x), E(x, v_0)\}$.
$c_2 = \{U_0(x)\}, \neg E(x, v_0), E(x, v_1)\}$.
$c_3 = \{U_1(x), E(x, v_0), \neg E(x, v_1), E(x, v_2)\}$.

Diagonal Antichains

An antichain of coding nodes in a coding tree of 1-types is **diagonal** if the branching degree is 2 and at each level of the tree there is at most one branching node or one coding node (never both).



Structural Properties corresponding to Diagonal Antichains

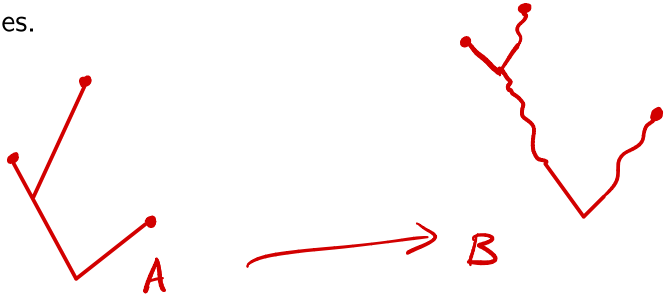
A a substructure of **K** is induced by a diagonal antichain iff

- **(antichain)** For each pair v_m, v_n of vertices in **A**, there is some $i < \min(m, n)$ such that v_m and v_n differ on some relation with v_i.
- **(diagonal)** Let v_k, v_ℓ, v_m, v_n be distinct vertices in **A** for which there is an $i < \min(k, \ell, m, n)$ such that i is least such that v_k and v_ℓ differ on some relation with v_i and i is least such that v_m and v_n differ on some relation with v_i. Then either v_k and v_m have the same relations over $\mathbf{K}_{i+1}$ or else v_k and v_n do. Moreover, v_i is not a vertex of **A**.

Similarity

Two substructures $\mathbf{A}, \mathbf{B}$ of $\mathbf{K}$ are **similar** if

- $\mathbf{A}$ and $\mathbf{B}$ are isomorphic as ordered structures, with the ordering inherited from the order of their vertices in $\mathbf{K}$.
- The increasing map from the vertices in $\mathbf{A}$ to the vertices in $\mathbf{B}$ induces a tree map between the subtrees induced by their coding nodes in $\mathbb{S}(\mathbf{K})$ which preserves splitting nodes, the left-right direction of their immediate successors, and relative lengths of nodes.



Remarks

- In order to obtain a theorem of the form

$$\mathbf{K} \to_* (\mathbf{K})^{\mathbf{K}}$$

the existence of big Ramsey degrees seems to necessitate restricting to a subspace of $\binom{\mathbf{K}}{\mathbf{K}}$ in which all subcopies of $\mathbf{K}$ have the same similarity type.

- In order to directly recover big Ramsey degrees, it is necessary to work with diagonal antichains.

Spaces of Diagonal Coding Antichains

Fix $\mathbf{K}$ and let $\mathbb{D}$ be a diagonal antichain of coding nodes in $\mathbb{S}(\mathbf{K})$ representing a subcopy of $\mathbf{K}$.

$\mathcal{D}(\mathbb{D}) = \{M \subseteq \mathbb{D} : M \sim \mathbb{D}\}$.

For $M \in \mathcal{D}(\mathbb{D})$, let $\mathbf{M} = \mathbf{K} \restriction \{v_i : c_i \in M\}$.

$\mathbf{D} = \mathbf{K} \restriction \mathbb{D}$ and

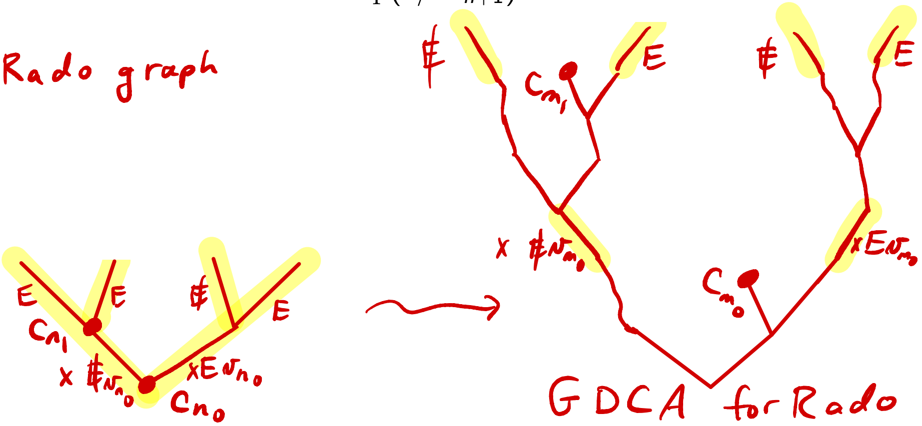
$$\mathbf{K}(\mathbf{D}) = \{\mathbf{M} : M \in \mathcal{D}(\mathbb{D})\}, \text{ a subspace of } \binom{\mathbf{K}}{\mathbf{K}}.$$

Identify $\mathbf{K}(\mathbf{D})$ with the subspace of $[\omega]^\omega$ via $\mathbf{M} \mapsto \{i \in \omega : v_i \in \mathbf{M}\}$.

Good Diagonal Coding Antichains for subcopies of **K**

A diagonal coding antichain $M \subseteq \mathbb{S}(\mathbf{K})$ is **good** if

- $\exists k$ such that $\forall n \geq k$, to each 1-type σ over $\mathbf{K}_{n+1}$ there corresponds a unique node $s \in M$ of length $|c_n^M| + 1$ such that
$$\mathrm{tp}(s/\mathbf{M}_{n+1}) \sim \sigma.$$



Infinite-Dimensional Ramsey Theory for SDAP$^+$ structures

Theorem (D.)

Let $\mathbf{K}$ be a Fraïssé structure satisfying SDAP$^+$ with finitely many relations of arity at most two. Let $\mathbb{D}$ be a good diagonal coding antichain, and $\mathbf{D} = \mathbf{K} \upharpoonright \mathbb{D}$. Then every Borel subset of $\mathbf{K}(\mathbf{D})$ is completely Ramsey.

Corollary

If $\mathbf{K}$ has a certain amount of rigidity, Axiom A.3(2) of Todorcevic also holds, so we obtain topological Ramsey spaces.

Corollary

We recover exact big Ramsey degrees via certain envelopes and the lower bound result of (Coulson–D.–Patel).

Proof Outline.

Uses outline of (D. 2019), which follows outline of Galvin–Prikry.

1. Show that all open sets are completely Ramsey.
2. Show that complements of Ramsey sets are completely Ramsey.
3. Show that completely Ramsey sets are closed under countable unions.

Since Todorcevic's Axiom A.3(2) usually fails for these spaces, we make up for it by proving a strengthened Pigeonhole Principle. This is where forcing arguments are used to do unbounded searches for homogeneous level sets.

Strong Pigeonhole Lemma Set-up.

Fix a good diagonal coding antichain $\mathbb{D}$; let $\mathcal{D} = \mathcal{D}(\mathbb{D})$.

We work with triples (A, B, k), where $A \in \widehat{\mathcal{AD}}$, $B \subseteq \widehat{\mathbb{D}}$, and $A \sqsubset B \subseteq A^+$.

Assume that all splitting nodes in A, B, D are not splitting predecessors in $\mathbb{D}$.

Case (a). $\max(r_{k+1}(\mathbb{D}))$ has a splitting node.

Case(b). $\max(r_{k+1}(\mathbb{D}))$ has a coding node.

Case (i). $k \geq 1$, $A \in \mathcal{AD}_k$, and $B = A^+$.

Case (ii). $k \geq 0$, $A \neq \emptyset$, each member of $\max(A)$ has exactly one extension in $\max(B)$, and $A = C \upharpoonright \ell$ for some $C \in \mathcal{AD}_{k+1}$ and $\ell < \ell_C$ such that $r_k(C) \sqsubseteq A$ and $B \sqsubseteq C$.

Strong Pigeonhole Lemma.

Lemma (Strong Pigeonhole)

Let (A, B, k) be as above satisfying one of Cases (a) and (b) and one of Cases (i) and (ii), and let $h : r_{k+1}[D, \mathbb{D}]^* \to 2$ be a coloring. Then there is an $M \in [D, \mathbb{D}]^*$ such that h is monochromatic on $r_{k+1}[B, M]^*$.

$[B, \mathbb{D}]^*$ is a loosening of $[B, \mathbb{D}]$.

When $B = r_n(M)$ for some $M \in \mathcal{D}$, then $[B, \mathbb{D}]^* = [B, \mathbb{D}]$.

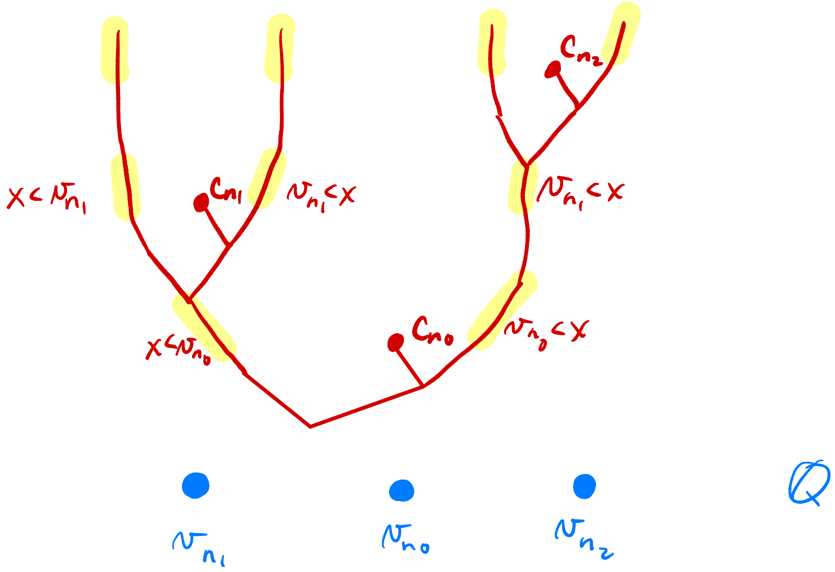
A subset $\mathcal{X} \subseteq \mathcal{D}$ is **CR*** if for each nonempty $[B, \mathbb{D}]^*$, there is an $M \in [B, \mathbb{D}]^*$ such that either $[B, M]^* \subseteq \mathcal{X}$ or else $[B, M]^* \cap \mathcal{X} = \emptyset$.

Proof Ideas

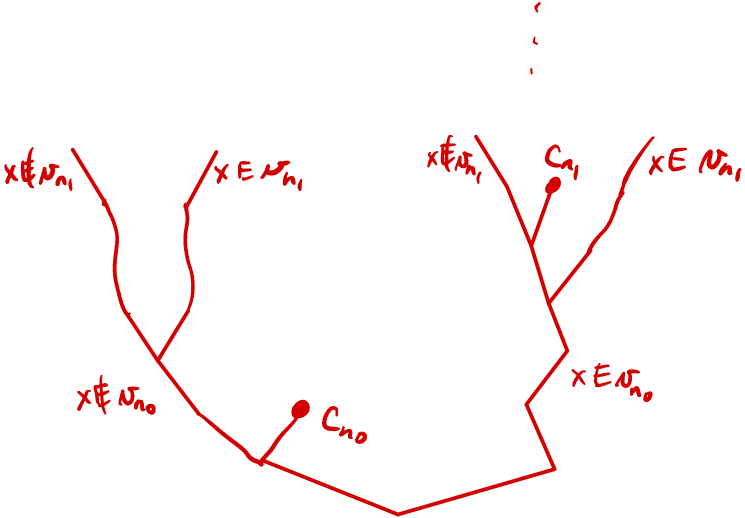
The proof uses a combination of ideas from

- The forcing proofs for diagonal coding trees in (D. 2017 and 2019) for Henson graphs, which uses three different forcings.
- Proof outline from (D. 2019) for ∞-dimensional Ramsey theory of the Rado graph.
- Ideas from (CDP 2020, Part I) for forcing on coding trees of 1-types.
- New ideas for forcing on diagonal coding antichains.

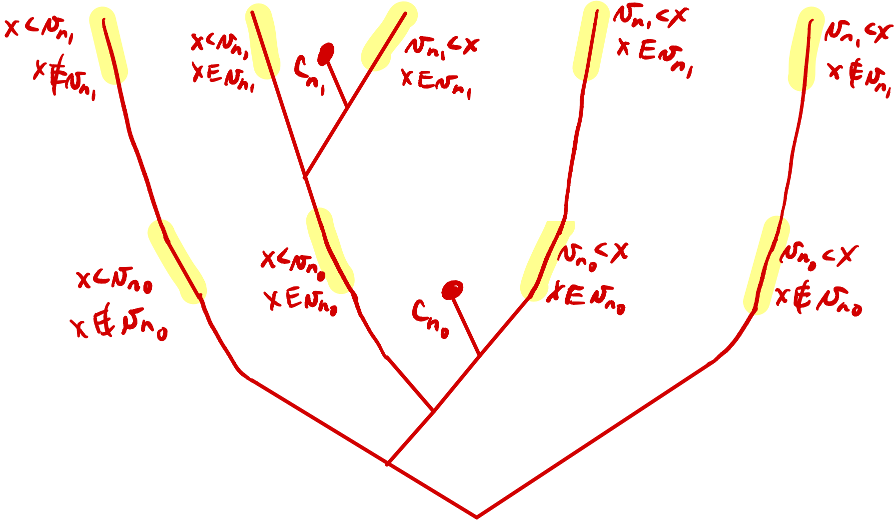
Picture for $\mathbb{Q}$



Picture for Rado graph



Picture for $\mathbb{Q}_{\mathbb{Q}}$



From the DAC's back to the Baire Space

Let $\theta : \mathbf{K}(\mathbb{D}) \to \mathcal{D}(\mathbb{D})$ denote the map which sends each $\mathbf{M} \in \mathbf{K}(\mathbb{D})$ to the subtree of $\mathbb{D}$ induced by the coding nodes representing the vertices in $\mathbf{M}$.

Then Borel sets in $\mathbf{K}(\mathbb{D})$ correspond with Borel sets in $\mathcal{D}(\mathbb{D})$.

Moreover, we obtain that every Borel subset of $\mathbf{K}(\mathbb{D})$ is completely Ramsey.

Recovery of exact big Ramsey degrees uses a canonical envelope construction and finitely many applications of the main Theorem.

Binary FAP

Theorem (D., Zucker)

Let $\mathcal{L}$ be a finite set relation symbols of arity at most two, and $\mathcal{F}$ be a finite set of finite irreducible $\mathcal{L}$-structures. Then for $\mathbf{K} =$ *Flim(Forb($\mathcal{F}$)), we have infinite-dimensional Ramsey theory which directly recovers exact big Ramsey degrees (BCDHKVZ 2021).*

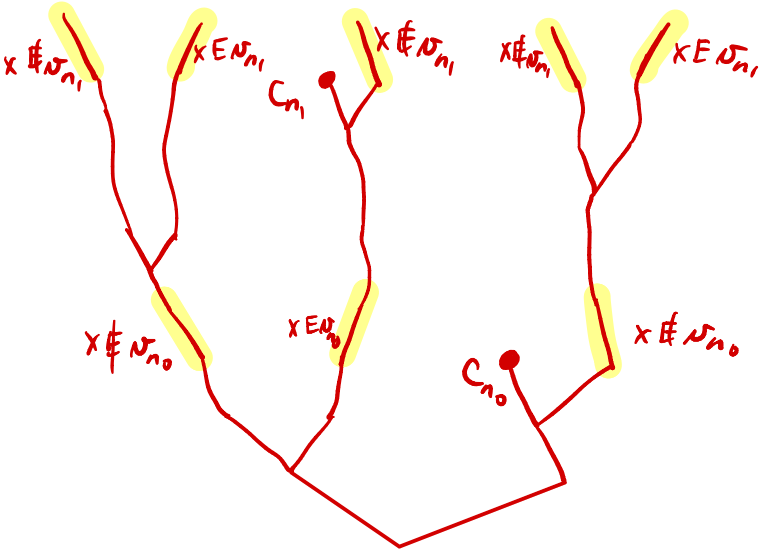
Examples. k-clique-free Henson graphs and homog. k-partite graphs.

Remarks.

(1) This theorem incorporates methods from (Zucker 2020) and the several forcing arguments of (D. 2017–2022) for forcing on diagonal coding trees.

(2) Our work does not reprove the lower bound result in (BCDHKVZ 2021).

Example: The triangle-free Henson graph $\mathcal{H}_3$



Ongoing Investigations

- Non-forcing proofs for forbidden substructures. (work of Hubička and ongoing with Balko, Chodounský, D., Konečný, Nešetřil, Vena, Zucker)
- Relations of arity 3 or more.
(work of Balko, Chodounský, Hubička, Konečný, Vena, and ongoing)
- Reverse Mathematics aspects.
(AMS Memoirs: Anglès d'Auriac, Cholak, Dzhafarov, Monin, Patey)
- Model Theoretic aspects.
(work of Coulson, D., Patel and questions of Džamonja and Zilber)
- Topological dynamics correspondence.
(question of Kechris–Pestov–Todorcevic and work of Zucker)
- Infinite dimensional Ramsey theory. (D., Zucker)

Thank You!