Low levels of the arithmetical hierarchy and computable reductions on ω

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Surveying results of many people.

Ceers (Σ^0_1 equivalence relations)

A lot of work has been focused on the structure of ceers, including:

- There is a universal degree, which apears naturally: Provable equivalence in PA, isomorphism of finite presentations of groups, word problems of some groups, equivalence relations where the classes are uniformly effectively inseparable.
- Ceers with finitely many classes form an initial segment $\mathcal{I}$.
- There are ceers which are not above $=^\omega$ (usually called Id). We call these *dark*. This is a failure of the analog of Silver's theorem.
- There are infinitely many ceers which are minimal over $\mathcal{I}$.
- We have some descriptions of when pairs of ceers have (or don't have) a join or a meet.
- Every degree has a strong minimal cover (some only 1, some countably many)

More ceers facts

- $\omega^{<\omega}$ embeds as an initial segment of the degrees (sending the empty string to Id).
- The degree structure of Ceers interprets $(\mathbb{N}, +, \cdot)$ and so the theory is as complicated as possible. Also, the degree structure of the Light ceers, also the degree structure of the Dark ceers. Also, each of these $/\mathcal{I}$.
- The collection of 1-dimensional ceers R_X for $X \subseteq \omega$ embeds the 1-degrees of (infinite) c.e. sets.

Definition (The Halting Jump operator on ceers)

Given a ceer X, define X' by $i \; X' \; j$ if and only if $\phi_i(i) \downarrow X \phi_j(j) \downarrow$.

- $X' \geq X$ for all X.
- $X' > Y'$ iff $X > Y$.
- $X' \equiv X$ if and only if X if universal.
- $X' \leq A \oplus B$ implies $X' \leq A$ or $X' \leq B$.

Co-ceers (Π^0_1-equivalence relations)

- There is a universal co-ceer π.
- The only ceer which is below a co-ceer is Id, and the ones with finitely many classes.
- Every co-ceer is light (i.e. above Id).

Everything about ceers relativizes (some care needed: Relativizations include $0'$-reductions).

- There is universal Σ_2^0-equivalence relation.
- There are dark ones.
- There are the 1-dimensional ones (closed downwards)

We haven't really considered what the halting jump looks like here. e.g., What are there other fixed points besides the universal ceer degree and the universal Σ_2^0-degree?

For any Δ_2^0-degree $\mathbf{d}$, the complete $\mathbf{d}$-ceer is a fixed-point. Are there any others? Is the universal ceer least among the fixed points?

Very little independent investigation here.

Π^0_n for $n \geq 2$

Many natural examples of things that correspond to ERs on 2^ω restricted to CE: $=^{ce} \equiv \mathrm{Id}^{\dagger} \in \Pi^0_2$, $E^{ce}_{set} \equiv \mathrm{Id}^{\dagger\dagger} \in \Pi^0_4$, $E^{ce}_3 \in \Pi^0_4$

Definition

For any E, let $iE^{+}j$ if and only if $[W_i]_E = [W_j]_E$.

Theorem

There is NO universal Π^0_n-equivalence relation.

In fact, for every Π^0_n-equivalence relation X, there is some Δ^0_n-equivalence relation which is not below X.

This is a constant foot-gun. The temptation to say that $=^{ce}$ is Π^0_2-universal is overpowering at times. Resist.

Why not?!?

Theorem

If X is a Π^0_2-equivalence relation, then there is some $Y \in \Delta^0_2$ so that $i \ X \ j$ iff $Y^{[i]} = Y^{[j]}$.

Now, the Ershov-Hierarchy essentially answers why there can't be a universal one. Consider the sequence:

$=^{ce}$ formed by letting Y be a universal c.e. set.

Next $=^{d-ce}$ formed by letting Y be a universal d-c.e. set.

$\vdots$

$=^{\alpha-ce}$ formed by letting Y be a universal α-c.e. set.

$\vdots$

By looking at where Y sits in the Ershov hierarchy, it's clear that these are co-final among Δ^0_2-equivalence relations.

Aside on $=^{\Sigma^0_n}$ and $\dot{+}$

Relativizing at higher levels, that same hierarchy looks like: $=^{\Sigma^0_3} < =^{d-\Sigma^0_3} < \cdots$.

Theorem

$\mathrm{Id}^{\dot{+}n} \equiv =^{\Sigma^0_{2n-1}}$.

Corollary

Every Σ^0_{2n-1} or Π^0_{2n-1} equivalence relation reduces to $\mathrm{Id}^{\dot{+}n}$.

Proof.

If X is Σ^0_{2n-1}, we provide a reduction of X to $=^{\Sigma^0_{2n-1}}$. Send n to $[n]_X$.
If X is Π^0_{2n-1}, send n to $\omega \setminus [n]_X$. □

Question

Is $\pi^{\dot{+}} \equiv =^{\Sigma^0_2}$?

Aside on $=^{\Sigma^0_n}$ and $\dot{+}$ 2

$\dot{+}$ doesn't preserve these difference hierarchies:

Question

For any Π^0_n-equivalence relation X, $X^{\dot{+}} \leq =^{\Sigma^0_{n+1}}$.

Proof.

Send i to $[W_i]_X$. □

Question

We can ask about what the high Π^0_n-equivalence relations are. This has been looked at for the ceers with some surprising answers, but not even at Π^0_1.
Is $=^{ce}$ the least Π^0_2-equivalence relation X so that $X^{\dot{+}} \equiv =^{\Sigma^0_3}$?
Do they all have that jump?

So why is there a Π^0_1-universal?

Theorem

For every Π^0_1 relation (*not assumed transitive*) E, there is a Δ^0_1 set X and a partial computable function f so that *if* E is an equivalence relation, then $i \mathrel{E} j$ iff $X^{[f(i)]} = X^{[f(j)]}$.

Proof.

At every s, we determine $X(\langle n, m\rangle)$ for $n, m \leq s$. Let $t_0 = 0$ and let t_{n+1} be the first stage $> t_n$ where E looks transitive on $[0, n+1]$. If E is transitive, then this is an infinite sequence of stages, and $f : n \mapsto t_n$ will be our reduction. When s is not a t_n-stage for some n, we do nothing much in coding X – make no differences. Put 0 on all new inputs.
Otherwise, code the highest-priority split – use transitivity to make all the coding columns look okay. □

We could do this for Π^0_2-relations, but the reduction function f would also be Δ^0_2, so we wouldn't get computable reduction.

Here lie some natural ERs on c.e. sets:

$$E_0^{ce} \equiv E_1^{ce} \equiv E_2^{ce} \equiv \text{the } \Sigma^0_3\text{-universal degree}$$

Definition

$iE_0^{ce}j$ iff $W_i =^* W_j$
$iE_1^{ce}j$ iff for all but finitely many n, $W_i^[n] = W_j^[n]$
$iE_2^{ce}j$ iff $\Sigma_{n \in A \triangle B} \frac{1}{n} < \infty$

The pattern seems to be that almost any "natural" Σ^0_n-equivalence relation will collapse to being universal. Obviously, this doesn't happen at Π-levels.

Some classes within Σ^0_3-ERs, including the following two attempts to "effectivize" the class of countable borel equivalence relations (cbers).

Countable Borel equivalence relations?

Definition (Coskey, Hamkins, R. Miller (2012))

- The action of a computable group G acting on **CE** is computable in indices if there is computable α so that

$$W_{\alpha(g,e)} = g \cdot W_e.$$

  The induced orbit equivalence relation is denoted E^{ce}_G.
- E^{ce} is enumerable in indices if there is computable α so that, for all $i \in \omega$,

$$e \, E^{ce} \, i \Leftrightarrow (\exists n)(W_{\alpha(e,n)} = W_i).$$

The first here was a natural attempt to use the Feldman-Moore theorem to bring the idea of cbers to ERs on **CE**. The second attempt is similar, but using the Luzin-Novikov theorem.

Dichotomy for groups

Theorem

If G is a computable group acting on **CE** computably in indices, then either $E_G^{ce} \equiv E_0^{ce}$ or $E_G^{ce} \equiv =^{ce}$

First, we showed that any group acting on **CE** computably in indices is actually acting via a permutation on ω. Still, there are several computable subgroups of S_∞ to consider.
The prototypical examples to consider come down to the following cases:

- Let G be all finite permutations of ω.
- Let $\mathbb{Z}$ act on ω by shifting.
- Let G be generated by $(0,1)(2,3,4)(5,6,7,8)\cdots$.

Having shown these were all Σ_3^0-complete, we realized that we had enough tricks to prove the same for any infinite $G \subseteq S_\infty$.

Non-dichotomy for enumerations

Theorem

There are infinite chains and antichains of ERs which are enumerable in indices between $=^{ce}$ and E_0^{ce}.

Simple construction for chains.

For $X \subsetneq \omega$, let $F(X)$ be the least element in X^c.
Let iR_nj if and only if $W_i = W_j$ or $0 \in W_i \cap W_j$ and $F(W_i) \equiv F(W_j) \mod n$.
Note that $=^{ce}$ reduces to R_n by sending W_i to $W_i + 1$. Among c.e. sets which contain 0, there are $n+1$ classes depending $F(W_i) \bmod n$ OR $F(W_i) = \infty$. The last one is Π^0_2-complete, while the others are Σ^0_2-complete. By counting the number of properly Σ^0_2-classes, you can show $R_{n+1} \not\leq R_n$. □

Some questions about enumerable in indices ERs

Our examples are all Δ^0_3. Can there be a properly Σ^0_3, but not universal, ER which is enumerable in indices?

Also, there is a Δ^0_2 enumerable in indices ER: E_{min}, and a Π^0_2 which is below $=^{ce}$: E_{max}.

Can there be a Σ^0_2 one which is not Δ^0_2. More generally, can there be any Σ^0_2 quotient of $=^{ce}$ which is not Δ^0_2?

Uniform enumeration in indices

Can the Lusin-Novikov direction be salvaged by demanding more uniformity from the enumerations?

Definition

E^{ce} is *uniformly* enumerable in indices if there is a computable α so that for all $i \in \omega$,

$$e\, E^{ce}\, i \Leftrightarrow (\exists n)(W_{\alpha(e,n)} = W_i).$$

and whenever $W_e = W_i$, $W_{\alpha(e,n)} = W_{\alpha(i,n)}$.

Note that you expect this if the operation $W_i \mapsto W_{\alpha(i,n)}$ is really an operation on sets (i.e., is independent of the enumeration).

Observation

E^{ce} is uniformly enumerable in indices if and only if it is the orbit equivalence of a computable action of a monoid M on **CE**.

Thank you

for your attention, comments and contributions!