Optimal Oracles for Point-to-Set Principles

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Fractal Geometry

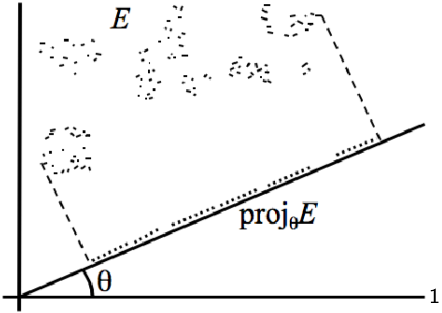
- Size of small/irregular **sets**
- Fractal dimensions - Hausdorff, packing, etc.

Algorithmic Randomness

- Inherent randomness of **points** (binary sequences)
- Computability theory

We can use computability theory to solve problems in **classical** fractal geometry

Orthogonal Projections



In the plane, we parameterize orthogonal projections with the angle the line makes with the x-axis, so

$$p_\theta : \mathbb{R}^2 \to \mathbb{R}$$
$$p_\theta(x, y) = x \cos\theta + y \sin\theta.$$

[1] Kenneth Falconer, Sixty years of fractal projections

Marstrands Projection Theorem

Theorem (Marstrand '54)

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_H(E) = s$. Then for almost every $\theta \in (0, 2\pi)$,

$$\dim_H(p_\theta E) = \min\{s, 1\}.$$

- Any subset of a line has Hausdorff dimension at most 1.
- Lipschitz functions (like p_θ) cannot increase the Hausdorff dimension of a set.
  - MPT shows that for a.e. angle, $\dim_H(p_\theta E)$ is maximal.
- Testing ground for new techniques.
- Useful in many different problems in fractal geometry.
- Wellspring of generalizations and open problems.

Marstrand's Projection Theorem

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Question: How necessary is the analytic assumption?

- Davies proved that, assuming CH, there are sets for which MPT does **not** hold.
- T. Slaman and S. (unpublished) proved that, assuming $V = L$, there are Π^1_1 counterexamples to MPT.
- N. Lutz and S. proved that, if $\dim_H(E) = \dim_P(E)$, then for almost every $\theta \in (0, 2\pi)$,

$$\dim_H(p_\theta E) = \min\{s, 1\}.$$

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In this talk, we present the notion of *optimal oracles*.

- If $E \subseteq \mathbb{R}^n$ has optimal oracles, then MPT holds for E.
- Every analytic set has optimal oracles.
- Every regular set E ($\dim_H(E) = \dim_P(E)$) has optimal oracles.
- Assuming CH, there are sets without optimal oracles.
- Assuming $V = L$, there are Π^1_1 sets without optimal oracles. (Slaman and S., unpublished)

Effective Dimensions of Points

Let $x \in \mathbb{R}$ and $r \in \mathbb{N}$. The *Kolmogorov complexity of x at precision r* is

$$
\begin{aligned}
K_r(x) &\approx \text{ Kolmogorov complexity of the first r bits of binary expansion of } x \\
&\approx \text{ length of shortest program outputting the first r bits} \\
&\qquad\qquad\qquad \text{of the binary expansion of } x \\
&\approx \text{ number of bits to specify the closet dyadic to } x \text{ at precision } 2^{-r}
\end{aligned}
$$

- Can generalize this to $\mathbb{R}^n$ in the natural way.
- $0 \leq K_r(x) \leq nr$

Effective Dimensions of Points

Definition (Lutz '03, Mayordomo '03)

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *(effective Hausdorff) dimension of* x is

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r}.$$

Definition (Athreya et al. '07, Lutz and Mayordomo '08)

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *(effective) strong dimension of* x is

$$\operatorname{Dim}(x) = \limsup_{r\to\infty} \frac{K_r(x)}{r}.$$

- $0 \leq \dim(x) \leq \operatorname{Dim}(x) \leq n$ for every $x \in \mathbb{R}^n$.

The Point-to-Set Principle

Theorem (J. Lutz and N. Lutz, '16)

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x), \textit{ and}$$

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \operatorname{Dim}^A(x).$$

- The Hausdorff and packing dimensions of a *set* are characterized by the corresponding dimension of the *points* in the set, relative to a Hausdorff and packing oracle.
- Allows us to use algorithmic techniques to answer questions in fractal geometry.

Optimal Oracles

Let $x \in \mathbb{R}^n$, $B \subseteq \mathbb{N}$ and $\epsilon > 0$. We say that B is *ϵ-low for x* if, for almost every $r \in \mathbb{N}$,

$$K_r^B(x) \geq K_r(x) - \epsilon r.$$

Definition (Stull '21)

Let $E \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{N}$. We say that A is *Hausdorff optimal* for E if the following conditions are satisfied.

1. A is a Hausdorff oracle for E.
2. For every $B \subseteq \mathbb{N}$ and every $\epsilon > 0$ there is a point $x \in E$ such that $\dim^{A,B}(x) \geq \dim_H(E) - \epsilon$ and B is ϵ-low for x relative to A. That is,
$$K_r^{A,B}(x) \geq K_r^A(x) - \epsilon r.$$

Optimal Oracles

Definition (Stull '21)

Let $E \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{N}$. We say that A is *Hausdorff optimal* for E if the following conditions are satisfied.

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A is an optimal Hausdorff oracle for E if

- $\dim_H(E) = \sup_{x \in E} \dim^A(x)$.
- Given any oracle B, can find a high dimension point in $x \in E$ such that B contains almost no information about x.

Optimal Oracles

Lemma (S. '21)

If E is regular, i.e. $\dim_H(E) = \dim_P(E)$, then E has optimal oracles.

1. Let A_1 and A_2 be Hausdorff and packing oracles of E. Let A be the join of A_1, A_2.
$$\dim_H(E) = \sup_{x \in E} \dim^A(x) \text{ and } \dim_P(E) = \sup_{x \in E} \operatorname{Dim}^A(x)$$
2. Let $\epsilon > 0$ and $B \subseteq \mathbb{N}$. By the point-to-set principle, there is an $x \in E$ such that
$$\dim^{A,B}(x) > \dim_H(E) - \epsilon/4.$$
3. Hence, for all sufficiently large $r \in \mathbb{N}$,
$$\begin{aligned}(\dim_H(E) - \epsilon/2)\, r &\leq K_r^{A,B}(x) \\ &\leq K_r^A(x) + O(\log r) \\ &\leq (\dim_H(E) + \epsilon/2)\, r.\end{aligned}$$

Optimal Oracles

Theorem (S. '21)

If E is analytic then E has optimal oracles.

1. It suffices to show this for compact E.
2. Standard facts from geometric measure theory give the existence of "nice" Borel measure μ supported on E.
3. We encode μ into an oracle, and join this with a Hausdorff oracle for E.
4. Use a generalization of Levin's coding theorem to prove that, for every $B \subseteq \mathbb{N}$, B is low for x, for almost every x.
   - Proves a stronger statement: For every $B \subseteq \mathbb{N}$, for μ-a.e. $x \in E$,
     $$K_r^A(x) \geq K_r^{A,B}(x) - O(\log r).$$

Optimal oracles

Know that a set $E \subseteq \mathbb{R}^n$ has optimal oracles if

- E is regular, or
- E is analytic.

Conditional results:

- (S. '21) Assuming CH, there are sets without optimal oracles.
- (Slaman and S.) Assuming $V = L$, there is a Π^1_1 set without optimal oracles.
- Assuming AD, **every** set has optimal oracles.

Optimal oracles

Theorem

If $E \subseteq \mathbb{R}^2$ has optimal oracles, then for a.e. $\theta \in [0, 2\pi)$

$$\dim_H(p_\theta E) = \min\{\dim_H(E), 1\}.$$

- Weakest known sufficient conditions for MPT to hold.
- Frames the primary obstruction of MPT in terms of algorithmic information.

Optimal oracles in other contexts

- Most results in classical geometric measure theory assume that the set is analytic.
- Sometimes this is **not** necessary
  - N. Lutz theorem on slices and intersections
  - N. Lutz and S. theorem on Furstenberg sets
  - X. Guo and S. theorem on **packing** dimension of projections (unpublished).
- For certain problems, some assumption on the set is necessary.
- If $E \subseteq \mathbb{R}$ is a Borel subring, then $\dim_H(E) = 0$ or $E = \mathbb{R}$.
- If $E \subseteq \mathbb{R}^n$ is analytic and $\dim_H(E) > n/2$, then $\mu(\Delta E) > 0$.
  - Falconer's distance set problem, open for all $n \geq 2$.
- (Slaman and S.) Assuming $V = L$, there are Π^1_1 counterexamples. These counterexamples all very similar to the counterexample for optimal oracles.

The End

Thank you!