Geometric triviality in differentially closed fields

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The aim of this talk is to:

- Describe the most important open problem in the study of the theory DCF_0.
- Recall the ω-categoricity conjecture of D. Lascar and the corresponding counterexample of J. Freitag and T. Scanlon
- Describe recent developments around work placing their counterexample into a larger context.
- Point out the main open questions.

1. Refresher on DCF_0.

- The Trichotomy Theorem is arguably the deepest result in the study of DCF_0.

Zilber's Principle holds in DCF_0 (Hrushovski-Sokolovic, 1994)

Let Y be a strongly minimal set in $(\mathcal{U}, D) \models DCF_0$. Then exactly one of the following holds:

1. **(Field-like)** *Y is non-orthogonal to the algebraically closed subfield of constants $C_{\mathcal{U}} = \{y \in \mathcal{U} : D(y) = 0\}$; or*
2. **(Group-like)** *Y is non-orthogonal to a very special strongly minimal subgroup of an abelian variety not defined over $C_{\mathcal{U}}$; or*
3. *Y is* **(geometrically) trivial**.

- Except for (3), this gives a full classification of strongly minimal sets in DCF_0.

- We work in the language $L_D = (0, 1, +, \times, D)$ of differential rings.
- DF_0 denotes the theory of differential fields of characteristic zero:

A **differential field** (K, D) is a field K equipped with a derivation $D : K \to K$, i.e. an additive group homomorphism satisfying the Leibniz rule

$$D(x + y) = D(x) + D(y).$$

$$D(xy) = xD(y) + yD(x).$$

$$\left(\mathbb{C}(t), \frac{d}{dt}\right)$$

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- This theory can be quite wild: $(\mathbb{Q}, 0, 1, +, \times, D = 0)$ is an example of a differential field and so one gets non-computable definable sets.
- We look at existentially closed models.

- For each $m \in \mathbb{N}_{>0}$, associated with a differential field (K, D), is the ring of differential polynomials in m differential variables,

$$K\{\overline{X}\} = K[\overline{X}, \overline{X}', \ldots, \overline{X}^{(n)}, \ldots].$$

$\overline{X} = (X_1, \ldots, X_m)$

$X' = DX$

$D\overline{X} = (DX_1, \ldots, DX_n)$

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$$K\{\overline{X}\} = K[\overline{X}, \overline{X}', \dots, \overline{X}^{(n)}, \dots].$$

- If $f \in K\{\overline{X}\}$ is a differential polynomial, then the order of f, denoted $ord(f)$, is the largest n such that for some i, $X_i^{(n)}$ occurs in f.
- **Example:** $f(X) = (X')^2 - 4X^3 - tX$ is an example of a differential polynomial in $\mathbb{C}(t)\{X\}$ and $ord(f) = 1$.
- The analogue of algebraically closed fields in the differential context is defined as follows

Definition (Blum Axioms, 1968)

A differential field (K, D) is said to be differentially closed if for every $f, g \in K\{X\}$ such that $ord(f) > ord(g)$, there is $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$.

- DCF_0 is the theory obtained by adding to DF_0, the L_D-sentences describing that a differential field is differentially closed.

  It is a very tame theory.

Theorem (Blum, 1968)

The theory DCF_0 is complete & ω-stable; and has QE & EI.

- We fix throughout $(\mathcal{U}, D) \models DCF_0$ a saturated model.

Definition

A definable set $Y \subset \mathcal{U}^n$ is strongly minimal if Y is infinite and for every definable $X \subset Y$ either X or $Y \setminus X$ is finite.

- In DCF_0 strongly minimal sets determine, in a precise manner, the structure of all definable sets of finite Morley rank.

- In DCF_0, the model theoretic algebraic closure has the following nice characterization:

$$\begin{aligned} acl(A) &= \text{the field theoretic algebraic closure of the} \\ &\quad \text{differential field generated by } A. \\ &= \mathbb{Q}\left(a, a', a'', \ldots : a \in A\right)^{alg}. \end{aligned}$$

Fact

If Y is a strongly minimal set, then (Y, acl_Y) forms a pregeometry.

$acl_Y(A) = acl(A) \cap Y$

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General examples

- ACF_p: $acl =$ field theoretic algebraic closure.
- Vector spaces over a fixed field K: $acl = K$-span.

- The Trichotomy classifies non-trivial strongly minimal sets as essentially the above.

2. Trivial Pursuits

Definition

Let Y be a strongly minimal set. (Y, acl_Y) is (geometrically) trivial if for any $A \subset \mathcal{P}(Y)$ we have that

$$acl_Y(A) = \bigcup_{a \in A} acl_Y(a).$$

- Trivial strongly minimal sets have at most "binary structures", indeed

$$a \in acl_Y(A) \implies a \in acl_Y(b) \text{ for some } b \in A$$

- **Open problem:** Can trivial strongly minimal sets be classified?
  - If there were a strong structure theory, some of the strategy laid out by Hrushovki for certain diophantine problems might be possible.
  - Work on this problem has also lead to proof of important theorems in number theory/functional transcendence in a different direction.
  - New work of Rémi Jaoui that there is an abundance of such strongly minimal sets.

- Unfortunately, there is no general strategy to tackle the above problem.
- Can there even be a differential equation whose solution set has at most a "rich binary structure"?

Definition

A trivial strongly minimal set Y is ω-categorical if for any $y \in Y$ the set $acl_Y(y)$ is finite.

Old Conjecture (Lascar, 1976)

In DCF_0, geometric triviality $\implies$ ω-categoricity.

- Was there any evidence?

Theorem (Hrushovki, 1995)

The order 1 trivial strongly minimal sets are ω-categorical.

Conjecture (Sadly NOT True)

Let Y be a strongly minimal set in $(\mathcal{U}, D) \models DCF_0$. Then exactly one of the following holds:

1. *Y is Field-like; or*
2. *Y is Group-like; or*
3. *Y has no or little structure, i.e, is ω-categorical.*

- We now know from the work of Freitag and Scanlon that the conjecture is false.
- However, it is still possible that the conjecture is true for order 2 definable sets.

  Indeed, all known examples of order 2 trivial strongly minimal sets in DCF_0 are ω-categorical.

3. Freitag-Scanlon counterexample

- One would like to find a differential equation for which the solutions (a meromorphic function) satisfy rich binary algebraic relations!

Riemann Mapping Theorem

Let $D \subset \mathbb{C}$ be a simply connected domain that is not $\mathbb{C}$. Then there is a biholomorphic mapping $f : D \to \mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ is the complex upper half plane.

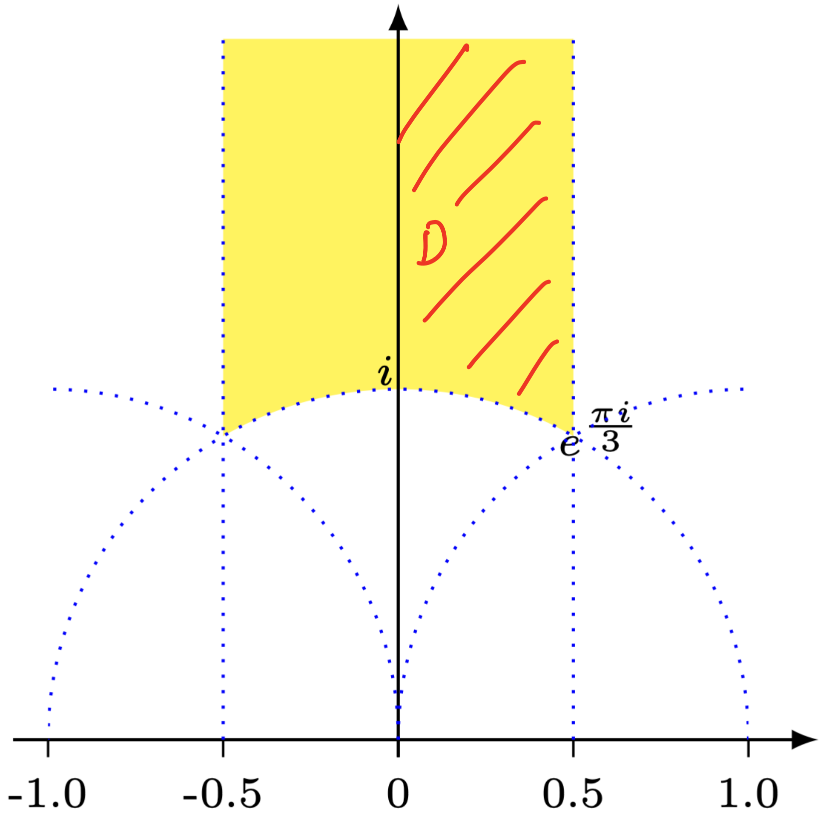
- Recall that we have $SL_2(\mathbb{R}) = \{A \in Mat_2(\mathbb{R}) : det(A) = 1\}$ and its action on $\mathbb{H}$ by linear fractional transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d} \in \mathbb{H}$$

$SL_2(\mathbb{Z})\backslash\mathbb{H}$

where $A \in SL_2(\mathbb{R})$ and $\tau \in \mathbb{H}$.

- The Freitag-Scanlon example focuses on the subgroup $SL_2(\mathbb{Z})$. In this case one has that the fundamental domain of action is



RMT

$j : D \to \mathbb{H}$

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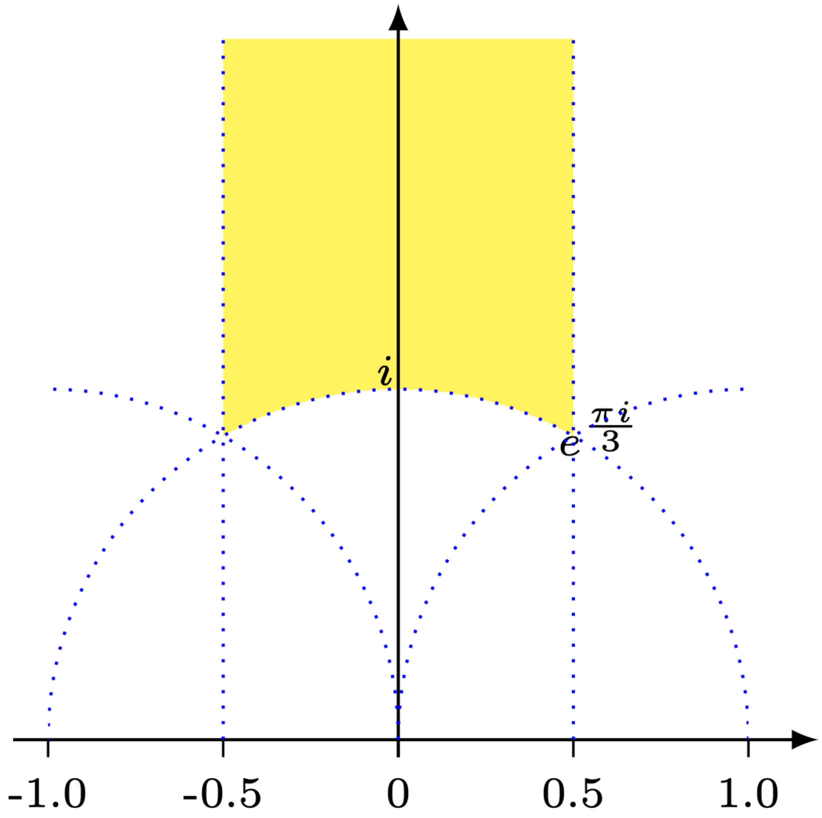

- We want to apply the RMT to the fundamental half domain.

$$
\begin{array}{llll}
\text{RMT} & \Longrightarrow & \exists & j : D \to \mathbb{H} \\
 & \downarrow & & \\
 & \Longrightarrow & & j : SL_2(\mathbb{Z}) \setminus \mathbb{H} \to \mathbb{C} \\
 & \downarrow & & \\
 & \Longrightarrow & & j : \mathbb{H} \to \mathbb{C} \text{ is } SL_2(\mathbb{Z})\text{-automorphic,} \\
 & & & \text{so } j(g \cdot t) = j(t) \text{ where } g \in SL_2(\mathbb{Z})
\end{array}
$$

- The function j is called the modular j-function.

  It satisfies the 3rd order algebraic differential equation

$$
\underbrace{\left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2}_{S_{\frac{d}{dt}}(y) =} + \frac{y^2 - 1968y + 2654208}{2y^2(y-1728)^2} \cdot (y')^2 = 0 \qquad (\star_j)
$$

Theorem (Freitag-Scanlon, 2018)

The set defined by the differential equation $(\star_j)$ *is strongly minimal, geometrically trivial BUT not* ω*-categorical.*

- They use a deep theorem of Pila in functional transcendence theory called the Modular Ax-Lindemann-Weierstrass Theorem.
- Other ingredients
  1. Any solution can be taken to be of the form $j(g \cdot t)$ for $g \in \overset{G}{\cancel{S}}L_2(\mathbb{R})$.
  2. For $N \in \mathbb{N}$, there exist a modular polynomial $\Phi_N \in \mathbb{C}[X, Y]$ such that

$$\Phi_N(j(t), j(N \cdot t)) = 0.$$

$$N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{acl}(j(t)) \ni j(N \cdot t)$$

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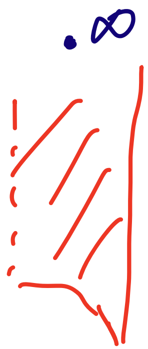
- Other ingredients
  1. Any solution can be taken to be of the form $j(g \cdot t)$ for $g \in SL_2(\mathbb{R})$.
  2. For $N \in \mathbb{N}$, there exist a modular polynomial $\Phi_N \in \mathbb{C}[X, Y]$ such that
$$\Phi_N(j(t), j(N \cdot t)) = 0.$$
- For a while this seemed to have shut the door on the possible classification of trivial strongly minimal sets.
- **Main question:** Is there a way to explain the existence of the modular polynomials?

3. The main results: the general context

- $SL_2(\mathbb{Z})$
- 

- j-function

- $\Gamma \subset SL_2(\mathbb{R})$ discrete $\underset{\text{defn}}{=}$ fuchsian gp.
- Γ is of first kind & genus 0

- RMT: $j_\Gamma : \mathbb{D} \to \mathbb{H}$

$$\Downarrow$$

$j_\Gamma : \mathbb{H} \longrightarrow \mathbb{C}$ invariant under Γ

$j(\gamma t) = j(t) \quad \gamma \in \Gamma$

- j_Γ uniformizer of Γ.

- The function j_Γ satisfies an order 3 ADE of Schwarzian type

$$S_{\frac{d}{dt}}(y) + \frac{1}{2}\sum_{i=1}^{r} \frac{1-{\alpha_i}^2}{(y-a_i)^2} + \sum_{i=1}^{r} \frac{A_i}{y-a_i} \cdot (y')^2 = 0 \qquad (\star_{j_\Gamma})$$

- We denote by X_Γ the set defined in $\mathcal{U}$ by equation $(\star_{j_\Gamma})$.

  As before, any solution in X_Γ can be taken to be of the form $j_\Gamma(g \cdot t)$.

$X_\Gamma = \{ j(gt) : g \in GL_2(\mathbb{R}) \}$

- The function j_Γ satisfies an order 3 ADE of Schwarzian type

$$S_{\frac{d}{dt}}(y) + \frac{1}{2}\sum_{i=1}^{r} \frac{1-\alpha_i{}^2}{(y-a_i)^2} + \sum_{i=1}^{r} \frac{A_i}{y-a_i} \cdot (y')^2 = 0 \qquad (\star_{j_\Gamma})$$

- We denote by X_Γ the set defined in $\mathcal{U}$ by equation $(\star_{j_\Gamma})$.

  As before, any solution in X_Γ can be taken to be of the form $j_\Gamma(g \cdot t)$.

Theorem (Casale-Freitag-N, 2020)

The definable set X_Γ is strongly minimal and geometrically trivial

- This amounts to a proof of an old conjecture of Painlevé (1895) about the irreducibility of equation $(\star_{j_\Gamma})$.
- Our proof is general and only depends on Γ (not on j_Γ) and in particular gives a new proof for $SL_2(\mathbb{Z})$.
- What about ω-categoricity?

- **Arithmeticity**: An important dividing line in group theory.

  Given any finite dimensional linear representation $SL_2(\mathbb{Z})$

  $$\rho : SL_2(\mathbb{R}) \to GL_n(\mathbb{R}).$$

  Let $G = \rho(SL_2(\mathbb{R})) \cap GL_n(\mathbb{Z})$.

- **Arithmeticity**: An important dividing line in group theory.

  Given any finite dimensional linear representation

$$\rho : SL_2(\mathbb{R}) \to GL_n(\mathbb{R}).$$

  Let $G = \rho(SL_2(\mathbb{R})) \cap GL_n(\mathbb{Z})$. Then $\Gamma = \rho^{-1}(G)$ is a Fuchsian group.

Definition

All subgroups of $SL_2(\mathbb{R})$ obtained this way and their subgroups of finite index are called arithmetic Fuchsian groups.

- Arithmetic Fuchsian groups play an important role in number theory and the quotients $\Gamma \setminus \mathbb{H}$ are known as (genus 0) Shimura Curves.

Theorem (Casale-Freitag-N, 2020)

The set X_Γ is not ω-categorical if and only if Γ is arithmetic.

Arithmeticity $\Rightarrow$ rich binary structure: fix $\Gamma < SL_2(\mathbb{R})$

Theorem (Poincaré): If j_1 & j_2 two automorphic uniformizer for Γ $\Rightarrow$ j_1, j_2 are alg dependent $/\mathbb{C}$

Fact: If $\Gamma_1 \subset \Gamma$ is of finite index then j_Γ is also uniformizer for Γ_1

If $g \in SL_2(\mathbb{R})$ s.t $\Gamma_g = \Gamma \cap (g\Gamma g^{-1})$ has finite index in both

$\Rightarrow$ $j(t)$ & $j(g^{-1}t)$ uniformizer for Γ_g

$\Rightarrow$ $\underbrace{j(t) \;\&\; j(g^{-1}t)}_{\text{alg dependent over } \mathbb{C}}$

$\Gamma \subset \mathrm{Comm}(\Gamma) = \{g \in SL_2(\mathbb{R}) : \Gamma_g \text{ has finite index in } \Gamma \text{ \& } g\Gamma g^{-1}\}$

Thm (Margulis): Γ is arithmetic iff Γ has infinite index in $\mathrm{Comm}(\Gamma)$.

Questions and speculations

Major Challenge

In DCF_0, does every non-ω-categorical strongly minimal set "arise" from an arithmetic Fuchsian group?

- In other words, is the following restatement of Lascar's Conjecture true?

Conjecture

Let Y be a strongly minimal set in $(\mathcal{U}, D) \models DCF_0$. Then exactly one of the following holds:

1. *Y is Field-like; or*
2. *Y is Group-like; or*
3. *Y arise from an arithmetic Fuchsian group; or*
4. *Y has no or little structure, i.e, is ω-categorical.*

Questions and speculations

Theorem (Freitag-Scanlon, 2018)

The set defined by the differential equation for the j-function is strongly minimal, geometrically trivial BUT not ω-categorical.

- Recall that we obtain non-ω-categoricity because:
  For $N \in \mathbb{N}$, there exist a modular polynomial $\Phi_N \in \mathbb{C}[X, Y]$ such that
  $$\Phi_N(j(t), j(N \cdot t)) = 0.$$
- The only method we have to detect the modular polynomial is via Seidenberg's Embedding theorem.
- The same is true for the other Fuchsian groups.
- **Interesting question:** Is there a proof of non-ω-categoricity of these equations “algebraically"?

Thank you very much for your attention.