Some (extra)ordinary equivalences in reverse mathematics

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Outline

1. Reverse mathematics
2. Two theorems by Rival and Sands
3. A variant and a new theorem

Equivalences

> *Reciprocation of premisses and conclusion is more frequent in mathematics [than in] dialectical disputations*
> *(Aristotle, Posterior Analytics, 78a10)*

Aristotle probably had in mind equivalences such as "a triangle has two congruent sides if and only if it has two congruent angles".

In modern mathematics we find many equivalences between statements, such as the one between the axiom of choice and Zorn's lemma on the basis of ZF.

All results of this sort can be called "reverse mathematics".

In a stricter sense, the term reverse mathematics applies to research carried out in the context of subsystems of second order arithmetic.

Mathematics in second order arithmetic

The idea that in (subsystems of) second order arithmetic it is possible to state and prove many significant mathematical theorems goes back to Hermann Weyl, Hilbert and Bernays.

The systematic search for the subsystems of second order arithmetic which are sufficient and necessary to prove these theorems was started by Harvey Friedman around 1970, and pursued by Steve Simpson and many others.

Subsystems of second order arithmetic

1. RCA_0: algebraic axioms for the natural numbers, $\mathbf{\Sigma}^0_1$-induction, and comprehension for $\mathbf{\Delta}^0_1$ formulas ω^ω
2. WKL_0: RCA_0+ König's lemma for binary trees ω^ω
3. ACA_0: comprehension extended to arithmetical formulas ε_0
4. ATR_0: ACA_0+ defin. by arithmetical transfinite recursion Γ_0
5. $\mathbf{\Pi}^1_1$-CA_0: comprehension extended to $\mathbf{\Pi}^1_1$ formulas $\Psi_{\Omega_1}(\Omega_\omega)$

RCA_0 is the base theory for reverse mathematics: it allows the development of "computable mathematics".
REC is the minimal ω-model of RCA_0.

RCA_0 and WKL_0 are $\mathbf{\Pi}^0_2$-conservative over PRA.

ACA_0 is conservative over PA.
ARITH is the minimal ω-model of ACA_0.

The big five of reverse mathematics

RCA_0, WKL_0, ACA_0, ATR_0, and $\mathbf{\Pi}^1_1$-CA_0 have been claimed to correspond to different approaches to the foundations of mathematics.

They can also be viewed as assertions about the existence of more and more incomputable sets and so are connected to computability theory.

The wealth of results showing their equivalence with mathematical theorems led to the terminology the big five. (when did this name originate? In Simpson's book they are called 'the five basic systems'; Montalbán's 2011 open questions paper uses it, but Antonio denies inventing it)

Induction

Induction plays an important role in reverse mathematics.

Today I will mention only the $\mathbf{\Sigma}^0_2$-induction scheme, that is

$$\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1)) \to \forall n\, \varphi(n)$$

for all $\mathbf{\Sigma}^0_2$ formulas φ.

RCA_0, WKL_0 and RT^2_2 do not prove $\mathbf{\Sigma}^0_2$-induction, which is provable in ACA_0.

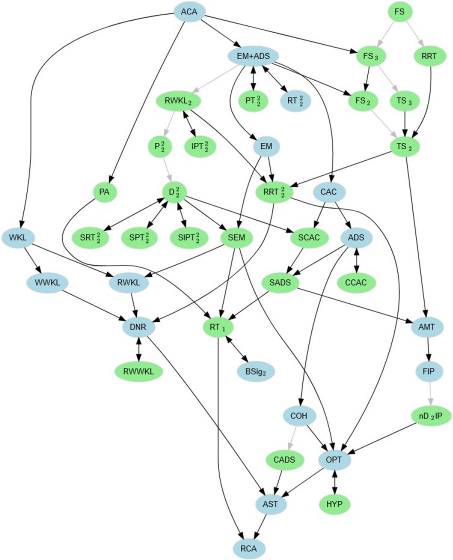
The zoo

In 1995 Seetapun showed that Ramsey Theorem for pairs and two colors RT^2_2 does not imply ACA_0.
It was already known that WKL_0 does not prove RT^2_2.
In 2012 Liu showed that RT^2_2 does not imply WKL_0.

After Seetapun's result, many statements provable in ACA_0 and incomparable with WKL_0 have been discovered.

The five-levels building of XXth century reverse mathematics is now much more complex, with lots of different beasts: the zoo of XXIst century reverse mathematics.

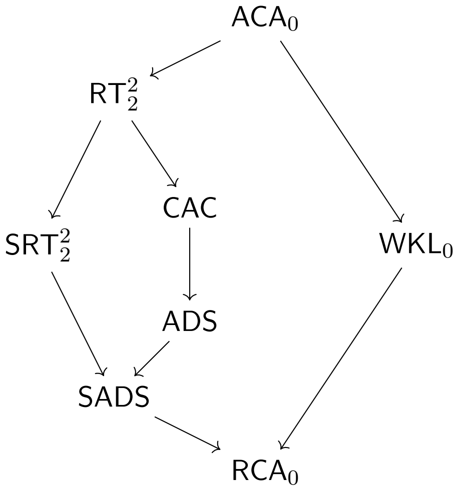
A portion of the zoo (from Ludovic Patey's website)



Five beasts in the zoo

- RT^2_2 Ramsey Theorem for Pairs: every infinite graph contains either an infinite clique or an infinite independent set
- SRT^2_2 Stable Ramsey Theorem for Pairs: RT^2_2 restricted to graphs such that every vertex has either finitely many or cofinitely many neighbors
- CAC Chain-Antichain: RT^2_2 restricted to comparability graphs
- ADS Ascending Sequence-Descending Sequence: every infinite linear order has either an infinite ascending sequence or an infinite descending sequence
- SADS Stable Ascending Sequence-Descending Sequence: ADS restricted to linear orders such that every element has either finitely many or cofinitely many predecessors

Relationships between the five beasts and three of the big five



Other zoos

The zoo phenomena is not peculiar to subsystems of second order arithmetic.

The study of weak forms of the Axiom of Choice and the relationships between them has a long tradition in set theory: the 1998 monograph *Consequences of the axiom of choice* by P. Howard and J. E. Rubin consists of a catalog of 383 forms of the Axiom of Choice and of their equivalent statements.

Connected to the book there was also a web site (apparently no longer maintained) producing zoo-like tables.

Plane geometry and weak logics are other areas where a zoo phenomena appears.

The zoo and ordinary mathematics

According to the original approach, the goal of reverse mathematics is to establish which axioms are required:

- to prove theorems of ordinary mathematics;
- to show that different definitions of the same ordinary mathematics concept are equivalent.

It is out of discussion that Ramsey Theorem for Pairs is a theorem of ordinary mathematics, but it is difficult to convince a mathematician outside logic that the same can be said of Chain-Antichain or of Ascending Sequence-Descending Sequence.

Although the study of zoo is interesting for its own sake (and useful to obtain information about the first-order consequences of various axioms), results that connect directly to ordinary mathematics are important (at least to me).

The theorem about graphs

Theorem (Rival-Sands 1980)

Every infinite graph G contains an infinite subset H such that every vertex of G is adjacent to precisely none, one, or infinitely many vertices of H.

Theorem (Fiori Carones-Shafer-Soldà 2022)

Over RCA_0, the previous statement is equivalent to ACA_0.
If we restrict the conclusion to vertices in H then the statement is equivalent to RT^2_2.

In the Weihrauch lattice, the problem arising from the Rival-Sands theorem is equivalent to the infinite parallelization of the problem arising from RT^2_2.

The theorem about partial orders

Theorem (Rival-Sands 1980)

Every infinite partial order P of finite width contains an infinite chain C such that every element of P is either comparable with no element of C or with infinitely many elements of C.

P has finite width if there exists k such that all antichains in P have size at most k. The least such k is the width of P.
$C \subseteq P$ is a chain if the elements of C are pairwise comparable.

If we consider the comparability graph of P (i.e. the graph with set of vertices P such that there is an edge between x and y iff $x \leq_P y$ or $y \leq_P x$), the theorem asserts that P (whose sets of independent vertices have bounded size) contains an infinite clique C such that all vertices are neighbors either of infinitely many members of C or of no member of C.
Thus, for a very special class of graphs, we get a clique and the one case of the Rival-Sands theorem for graphs is excluded.

Dilworth's theorem

Theorem (Hirst 1987)

Over RCA_0 *the following are equivalent:*

1. WKL_0;
2. *for every* k*, every partial order of width* k *is the union of* k *chains;*
3. *every partial order of width* 2 *is the union of two chains.*

Theorem (Kierstead 1981)

RCA_0 *proves that for every* k*, every partial order of width* k *is the union of* $(5^k - 1)/4$ *chains.*

The statements we study

$C \subseteq P$ is $(0,\infty)$-homogeneous if every $p \in P$ is either comparable with no element of C or with infinitely many elements of C.

Definition

- RS_k every infinite partial order of width k has an infinite $(0,\infty)$-homogeneous chain
- $\mathsf{RS}_{<\infty}$ $\forall k\, \mathsf{RS}_k$
- $\mathsf{RS}^{\mathsf{CD}}_k$ every infinite partial order which is the union of k chains has an infinite $(0,\infty)$-homogeneous chain
- $\mathsf{RS}^{\mathsf{CD}}_{<\infty}$ $\forall k\, \mathsf{RS}^{\mathsf{CD}}_k$

In RCA_0 it is immediate that $\mathsf{RS}_k \to \mathsf{RS}^{\mathsf{CD}}_k$ and Kierstead's result implies that $\mathsf{RS}^{\mathsf{CD}}_{(5^k-1)/4} \to \mathsf{RS}_k$.

Therefore $\mathsf{RS}_{<\infty} \leftrightarrow \mathsf{RS}^{\mathsf{CD}}_{<\infty}$ over RCA_0.

Main results

Theorem (FCMSS)

1. *For every $k > 2$, over* RCA_0, RS_k *and* $\mathsf{RS}_k^{\mathsf{CD}}$ *are equivalent to* ADS.
2. *Over* RCA_0, $\mathsf{RS}_2^{\mathsf{CD}}$ *is equivalent to* SADS.
3. *Over* WKL_0, RS_2 *is equivalent to* SADS.
4. *Over* RCA_0, $\mathsf{RS}_{<\infty}$ *and* $\mathsf{RS}_{<\infty}^{\mathsf{CD}}$ *are equivalent to* $\mathsf{ADS}+$ $\mathbf{\Sigma}_2^0$*-induction.*

Some theorems of ordinary mathematics are equivalent to zoo beasts such as ADS and SADS!

Reversing to SADS

Lemma

RCA_0 *proves that* $\mathsf{RS}_2^{\mathsf{CD}}$ *implies* SADS.

Proof.

Let L be an infinite linear order such that every element has either finitely many or cofinitely many predecessors.
Let $P = L \times 2$ ordered by $(x, i) \leq_P (y, j)$ iff $x \leq_L y$ and $i \leq j$.
P is the union of two chains.
By $\mathsf{RS}_2^{\mathsf{CD}}$, let C be a $(0, \infty)$-homogeneous chain in P.
C cannot contain (x, i) and (y, i) such that x has finitely many predecessors and y cofinitely many predecessors (if $i = 0$ consider $(x, 1)$, if $i = 1$ consider $(y, 0)$).
For some i, $C \times \{i\}$ is infinite: all its elements have finitely many predecessors (or successors), hence it contains an infinite ascending (descending) sequence. □

Reversing to ADS

Lemma

RCA_0 *proves that* $\mathsf{RS}_3^{\mathsf{CD}}$ *implies* ADS.

Proof.

Let L be an infinite linear order.
Let $P = L \times 3$ ordered by $(x, i) \leq_P (y, j)$ iff $x \leq_L y$ and either $i = j$ or $j = 1$.
P is the union of three chains.
By $\mathsf{RS}_3^{\mathsf{CD}}$, let C be a $(0, \infty)$-homogeneous chain in P.
C cannot intersect both $L \times \{0\}$ and $L \times \{2\}$ because those two chains are pairwise incomparable.
Assume $C \subseteq (L \times \{0\}) \cup (L \times \{1\})$.
$C \cap (L \times \{1\})$ has no maximum (if $(x, 1) \in C$ consider $(x, 2)$).
If $C \cap (L \times \{1\}) \neq \emptyset$ we find an ascending sequence.
Otherwise $C \subseteq L \times \{0\}$: in this case C has no minimum (if $(x, 0) \in C$ consider $(x, 1)$) and we find a descending sequence. □

The original proof

The proof by Rival and Sands relies on the following maximality principle:

MMLC For every partial order P there is a max-less chain which is $\subseteq$-maximal among the max-less chains of P.

Here a chain C is max-less if it has no maximum element.

Theorem (FCMSS)

Over RCA_0, MMLC *(even restricted to linear orders) is equivalent to* $\mathbf{\Pi}^1_1$-CA_0.

Rival-Sands' proof of $\mathsf{RS}_{<\infty}$ cannot be carried out in $\mathbf{\Pi}^1_1$-CA_0, because for a partial order of width k uses MMLC $k+1$ times.

Sketch of the forward direction

To prove our main theorem we devised a completely new proof.

Assume that the partial order of finite width P has no $(0,\infty)$-homogeneous chain.

Write P as a finite union of (pairwise disjoint) chains $C_0, \dots, C_k$.

Some C_{i_0} is infinite: use ADS to find either an infinite ascending sequence or an infinite descending sequence A_0 in C_{i_0}.

We can assume A_0 is increasing.

We build an increasing sequence $A_1 \subseteq C_{i_1}$ for some $i_1 \neq i_0$ which witnesses that A_0 and its tails are not $(0,\infty)$-homogeneous.

We iterate the process until we find $j < \ell < k$ such that $i_j = i_\ell$, i.e. A_j and A_ℓ are contained in the same chain C_{i_j}.

Then either a tail of A_j or $A_j \cup A_\ell$ is $(0,\infty)$-homogeneous.

A variant

A chain C in the partial order P is $(0, \mathrm{cof})$-homogeneous if every $p \in P$ is either comparable with no element of C or is comparable with cofinitely many elements of C.

Rival and Sands showed that countable partial orders of finite width contain $(0, \mathrm{cof})$-homogeneous chains.

- $(0, \mathrm{cof})$-RS_k every infinite partial order of width k has an infinite $(0, \mathrm{cof})$-homogeneous chain
- $(0, \mathrm{cof})$-$\mathsf{RS}_k^{\mathsf{CD}}$ every infinite partial order which is the union of k chains has an infinite $(0, \mathrm{cof})$-homogeneous chain

A result

Theorem (FCMSS)

Over $\mathsf{RCA}_0 + \mathbf{\Sigma}^0_2$*-induction,* $(0, \mathrm{cof})$-RS_2 *and* $(0, \mathrm{cof})$-$\mathsf{RS}_2^{\mathsf{CD}}$ *are equivalent to* ADS.

We do not know whether $\mathbf{\Sigma}^0_2$-induction is necessary here.

We know only that $\mathbf{\Pi}^1_1$-CA_0 proves $(0, \mathrm{cof})$-$\mathsf{RS}_{<\infty}$.
$(0, \mathrm{cof})$-$\mathsf{RS}_{<\infty}$ is a $\mathbf{\Pi}^1_2$ sentence and cannot be equivalent to $\mathbf{\Pi}^1_1$-CA_0.

Also, we have no better results about $(0, \mathrm{cof})$-RS_k and $(0, \mathrm{cof})$-$\mathsf{RS}_k^{\mathsf{CD}}$ for $k > 2$.

A new theorem and its strength

Theorem (FCMSS)

Over RCA_0*, the statement "every infinite partial order* P *with no infinite antichains has an infinite* $(0, \infty)$*-homogeneous chain" is equivalent to* ACA_0*.*

Theorem (FCMSS)

$\mathbf{\Pi}^1_1$-CA_0 *proves "every infinite partial order* P *with no infinite antichains has an infinite* $(0, \mathrm{cof})$*-homogeneous chain".*

The end

Thank you for your attention!