On the Weihrauch degrees of the additive Ramsey theorems

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Represented spaces and problems

We briefly recall the main definitions that we need.

- A **represented space** is a pair $\mathscr{X} = (X, \delta_{\mathscr{X}})$ such that X is a set and $\delta_{\mathscr{X}} :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is a partial surjection. For $x \in X$, $p \in \mathbb{N}^{\mathbb{N}}$ with $\delta_{\mathscr{X}}(p) = x$ is said to be a **name** of x.
- Let $\mathscr{X}$ and $\mathscr{Y}$ be represented spaces. A relation $f \subseteq X \times Y$ is called a **partial multifunction**, or **problem**, between $\mathscr{X}$ and $\mathscr{Y}$. $\mathrm{dom} f$ and $f(x)$ are defined in the obvious way. We describe problems in terms of their **inputs** and their corresponding **outputs**.

RT^2_2	Input: a coloring $c : [\mathbb{N}]^2 \to 2$ Output: an infinite c-homogeneous set H

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- Let $f :\subseteq X \rightrightarrows Y$ be a partial multifunction between $\mathscr{X}$ and $\mathscr{Y}$. A function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a **realizer** for f (written $F \vdash f$) if, intuitively, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{F} & \mathbb{N}^{\mathbb{N}} \\ \downarrow{\scriptstyle \delta_{\mathscr{X}}} & & \downarrow{\scriptstyle \delta_{\mathscr{Y}}} \\ X & \overset{f}{\Longrightarrow} & Y \end{array}$$

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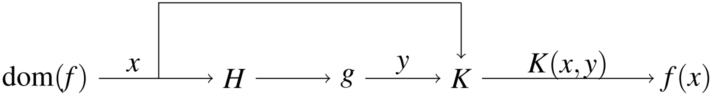
This perspective makes it possible to define **operations on problems**.

Weihrauch reducibility

Recall: we say that $H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a **Turing functional** if there is a computable $H^* : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ that approximates H.

f **Weihrauch reduces** to g ($f \leq_{\mathrm{W}} g$) if there are Turing functionals H, K such that the functional $p \mapsto K(\langle p, G(H(p))\rangle \vdash f$ whenever G is a realizer for g. Intuitively



Similarly, f **strongly Weihrauch reduces** to g if there are Turing functionals H, K as above, except that K **does not depend on** x.

$f \equiv_{\mathrm{W}} g$ if $f \leq_{\mathrm{W}} g$ and $g \leq_{\mathrm{W}} f$. Since this is an equivalence relation, we can define the lattice $\mathbb{W}$ of the Weihrauch degrees of problems.

Ramsey theorem over $\mathbb{Q}$

There are several ways to extend Ramsey theorem to the rationals. It is important (and easy) to note that not all of them hold. E.g., $\eta \longrightarrow (\eta)^1_{<\infty}$ holds, but it is not true that $\eta \longrightarrow (\eta)^2_2$.
Consider

$$c(x,y) = \begin{cases} 0 & \text{if } e_{\mathbb{Q}}(x) < e_{\mathbb{Q}}(y) \\ 1 & \text{otherwise.} \end{cases}$$

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But it holds that $\eta \longrightarrow (\aleph_0, \eta)^2_2$: see [FP17] for a reverse mathematical analysis of theses principles.

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For us, the following will be relevant.

- Given a coloring $c : [\mathbb{Q}]^n \to k$, we say that $]x,y[\subseteq \mathbb{Q}$ is a c-**shuffle** if there exists a finite partition $]x,y[= \bigcup_{i<m} H_i$ into dense subsets such that every H_i is c-homogeneous.

Lemma

$\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 \vdash$ *for every n and every coloring $c : \mathbb{Q} \to n$, there is a c-shuffle $]x,y[$, and this (clearly) implies $\eta \longrightarrow (\eta)^1_{<\infty}$.*

Shuffle	Input: a pair (k,c) such that $c : \mathbb{Q} \to k$ Output: $(C,(x,y))$ such that $]x,y[$ is a c-shuffle with colors C.

Ordered and additive Ramsey theorem for $\mathbb{Q}$

- Given a finite poset $(P, <_P)$ and a coloring $c : [\mathbb{Q}]^2 \to P$, we say that c is **ordered** if $x' \leq x < y \leq y'$ implies $c(x,y) \leq_P c(x',y')$.

$\mathsf{ORT}_{\mathbb{Q}}$ Input: a pair $((P, \leq_P), c)$ such that $c : [\mathbb{Q}]^2 \to P$ is ordered
Output: $(C, (x,y))$ such that $]x,y[$ is c-homogeneous for C.

- Given a finite semigroup $(S, \cdot)$ and a coloring $c : [\mathbb{Q}]^2 \to S$, we say that c is **additive** if, for every $x < y < z \in \mathbb{Q}$, $c(x,z) = c(x,y) \cdot c(y,z)$.

$\mathsf{ART}_{\mathbb{Q}}$ Input: a pair $((S, \cdot), c)$ such that $c : [\mathbb{Q}]^2 \to S$ is additive
Output: $(C, (x,y))$ such that $]x,y[$ is c-shuffle for $C \subseteq S$

- Given a coloring $c : [\mathbb{Q}]^n \to k$, we say that $]x,y[\subseteq \mathbb{Q}$ is a c-**shuffle** if there exists a finite partition $]x,y[= \bigcup_{i<m} H_i$ into dense subsets such that every H_i is c-homogeneous.

Ordered and additive Ramsey theorem for $\mathbb{N}$

- Given a finite poset $(P, <_P)$ and a coloring $c : [\mathbb{N}]^2 \to P$, we say that c is **right-ordered** if $x < y \le y'$ implies $c(x,y) \le_P c(x,y')$.

Notice: we could have defined "ordered" and "right-ordered" colorings for an arbitrary linear order $(X, <_X)$. In this case, notice that c ordered implies c right-ordered.

$\mathsf{ORT}_{\mathbb{N}}$ Input: a pair $((P, \le_P), c)$ such that $c : [\mathbb{N}]^2 \to P$ is ordered
Output: $H \subseteq \mathbb{N}$ such that H is infinite c-homogeneous.

- Given a finite semigroup $(S, \cdot)$ and a coloring $c : [\mathbb{N}]^2 \to S$, we say that c is **additive** if, for every $x < y < z \in \mathbb{N}$, $c(x,z) = c(x,y) \cdot c(y,z)$.

$\mathsf{ART}_{\mathbb{N}}$ Input: a pair $((S, \cdot), c)$ such that $c : [\mathbb{N}]^2 \to S$ is additive
Output: $H \subseteq \mathbb{N}$ such that H is infinite c-homogeneous

Although $\mathsf{ART}_{\mathbb{N}}$ seems rather boring (i.e., very trivial consequence of RT^n), we remark that it could be generalized to arbitrary limit ordinals ([She75]): namely, for every limit δ, every semigroup S with $|S| < \operatorname{cof} \delta$ and every additive $c : [\delta]^2 \to S$, there is a c-homogeneous H unbounded in δ.

A bit of history

These principles come from the study of MSO for the structure $(\mathbb{Q}, <)$:

- $\mathsf{ART}_{\mathbb{Q}}$ was first proved by Shelah ([She75]) to show that MSO for $(\mathbb{Q}, <)$ is decidable.
- $\mathsf{ART}_{\mathbb{N}}$ (or rather, its generalization) was proved by Shelah ([She75]) to show decidability of MSO for countable ordinals.
- Shuffle was introduced by Carton, Colcombet and Puppis ([CCP11]) to show a more general result.

Theorem (Kolodziejczyk and Pradic [Pra20])

$\mathsf{RCA}_0 \vdash \mathsf{ORT}_{\mathbb{Q}}$. *Moreover,*
$\mathsf{RCA}_0 \vdash \mathsf{Shuffle} \leftrightarrow \mathsf{ART}_{\mathbb{N}} \leftrightarrow \mathsf{ART}_{\mathbb{Q}} \leftrightarrow \mathsf{ORT}_{\mathbb{N}} \leftrightarrow \mathsf{I}\Sigma^0_2$.

Today, we will analyze their strength in the uniform setting.

Benchmark principles

$\mathsf{C}_{\mathbb{N}}$ Input: a sequence $e : \mathbb{N} \to \mathbb{N} \cup \{-1\}$ such that $\mathbb{N} \not\subseteq \operatorname{ran} e$
Output: $n \in \mathbb{N} \setminus \operatorname{ran} e$

We will mostly use the **total continuation** of $\mathsf{C}_{\mathbb{N}}$.

$\mathsf{TC}_{\mathbb{N}}$ Input: **any** closed set of $\mathbb{N}$ (i.e., any $e : \mathbb{N} \to \mathbb{N} \cup \{-1\}$)
Output: $n \notin \operatorname{ran} e$ if there is one, any $n \in \mathbb{N}$ otherwise

$\mathsf{TC}_{\mathbb{N}}$ was shown to be strongly related to $\mathsf{I}\Sigma^0_2$.

ECT Input: (k,f) such that $f : \mathbb{N} \to k$
Output: any $b \in \mathbb{N}$ such that $\forall x > b \exists y > x (f(x) = f(y))$

Theorem ([Dav+20])

- $\mathsf{ECT} \equiv_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}}$
- $\mathsf{RCA}_0 \vdash \mathsf{ECT} \leftrightarrow \mathsf{I}\Sigma^0_2$

Benchmark principles (cont.)

LPO	Input: a sequence $p \in 2^{\mathbb{N}}$ Output: 0 if $p = 0^{\mathbb{N}}$, 1 otherwise

We will be chiefly interested in LPO'.

f'	Input: a sequence $(x_i)_{i \in \mathbb{N}}$ converging to $x \in \mathrm{dom} f$ Output: an f-solution to x

An alternative formulation of LPO' will prove to be useful.

Lemma

$\mathsf{LPO}' \equiv_{\mathrm{sW}} \mathsf{IsFinite}$

where

IsFinite	Input: A sequence $p \in 2^{\mathbb{N}}$ Output: 0 if for finitely many i $p(i) = 0$, 1 otherwise

Relationship between LPO' and $\mathsf{TC}_{\mathbb{N}}$

It is easy to see that

- $\mathsf{LPO}' \not\leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$ by straightforward diagonalization.
- $\mathsf{C}_{\mathbb{N}} \not\leq_{\mathrm{W}} (\mathsf{LPO}')^*$, essentially because $\mathsf{C}_{\mathbb{N}}$ is not computed by problems with finite codomain, hence $\mathsf{TC}_{\mathbb{N}} \not\leq_{\mathrm{W}} (\mathsf{LPO}')^*$.

Lemma

For every $a, b \in \mathbb{N}$ and every single-valued $\mathsf{P} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with $\mathsf{P} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$, we have that

$$(\mathsf{TC}_{\mathbb{N}}^a \times (\mathsf{LPO}')^b) \star \mathsf{P} \leq_{\mathrm{W}} (\mathsf{TC}_{\mathbb{N}}^a \times (\mathsf{LPO}')^b) \times \mathsf{P}$$

$\mathsf{Q} \star \mathsf{P}$	Input: a pair (e, x), where $e \in \mathbb{N}^{\mathbb{N}}$ and $x \in \operatorname{dom} \mathsf{P}$ Output: (z, y) such that $y \in \mathsf{P}(x)$ and $z \in \mathsf{Q}(\Phi_e(\mathsf{P}(x)))$

The main point of the proof is that $\mathsf{LPO}'(x) = \mathsf{LPO}(w^\frown x)$ for every $w \in \mathbb{N}^*$, and similarly for $\mathsf{TC}_{\mathbb{N}}$ (not really, but almost: more on this later): P is solved by a finite-mind-changes computation, and by single-valuedness we know that $\mathsf{TC}_{\mathbb{N}}$ and LPO' are looking at the same solution.

The case of $\mathbb{Q}$: relationship between Shuffle and $\mathsf{ART}_{\mathbb{Q}}$

Shuffle	Input: a pair (k,c) such that $c:\mathbb{Q}\to k$ Output: $(C,(x,y))$ such that $]x,y[$ is a c-shuffle with colors C.

$\mathsf{ART}_{\mathbb{Q}}$	Input: a pair $((S,\cdot),c)$ such that $c:[\mathbb{Q}]^2\to S$ is additive Output: $(C,(x,y))$ such that $]x,y[$ is c-shuffle for $C\subseteq S$

It is immediate that Shuffle $\leq_{\mathrm{sW}}$ $\mathsf{ART}_{\mathbb{Q}}$

Sketch of the proof: suppose we are given $c:\mathbb{Q}\to k$, we define $f_c:[\mathbb{Q}]^2\to(k,\cdot_k)$ by setting $f_c(x,y)=c(x)$ for every $x<y$, and $a\cdot_k b=a$ for every $a,b<k$.

But it is **not** immediate to see whether $\mathsf{ART}_{\mathbb{Q}}\leq_{\mathrm{W}}$ Shuffle.

But notice that it is if $(S,\cdot)$ is a **group**: we fix $u<v\in\mathbb{Q}$ and define the coloring $\widetilde{c}:]u,v[\to S$ such that $\widetilde{c}(z)=c(z,v)$. Apply Shuffle to find $]x,y[$ that is $\widetilde{c}$-shuffle. Then $]x,y[$ is a c-shuffle as well: if $\widetilde{c}(w)=\widetilde{c}(z)$, notice that

$$c(w,z)\cdot c(z,v)=c(w,v)=c(z,v)\to c(w,z)=1_S.$$

Intervals and colors

Spoiler alert: Shuffle $\equiv_W$ $\mathsf{ART}_{\mathbb{Q}} \equiv \mathsf{TC}^*_{\mathbb{N}} \times (\mathsf{LPO}')^*$

It is practical to divide the problems Shuffle and $\mathsf{ART}_{\mathbb{Q}}$ into their **color part** and their **interval part**.

Recall that Shuffle

Input: a pair (k,c) such that $c:\mathbb{Q}\to k$
Output: $(C,(x,y))$ such that $]x,y[$ is a c-shuffle with colors C.

Lemma

- cShuffle $\equiv_W$ $\mathsf{cART}_{\mathbb{Q}} \equiv_W (\mathsf{LPO}')^*$
- iShuffle $\equiv_W$ $\mathsf{iART}_{\mathbb{Q}} \equiv_W \mathsf{TC}^*_{\mathbb{N}}$

Although it is not immediately obvious that Shuffle $\leq_W$ cShuffle $\times$ iShuffle (color and interval may refer to different solutions), one can combine the proofs above to obtain the Theorem.

cShuffle

Lemma

$\mathsf{IsFinite}^* \leq_W \mathsf{cShuffle}^*$

Sketch of the proof: clearly, $\mathsf{cShuffle} \times \mathsf{cShuffle} \leq_W \mathsf{cShuffle}$, hence it suffices to show that $\mathsf{IsFinite} \leq_W \mathsf{cShuffle}$.
Given $p \in \operatorname{dom} \mathsf{IsFinite}$, define $c_p : \mathbb{Q} \to 2$ as follows: $c_p\left(\frac{a}{b}\right) = p(b)$. Then, $0 \in \mathsf{cShuffle}(c_p)$ if and only if $p(b) = 0$ for infintely many b.

For the other direction, we give a finer analysis.

Lemma

Let $\mathsf{cShuffle}_n$ *be the restriction of* $\mathsf{cShuffle}$ *to colorings with exactly* n *colors. Then,* $\mathsf{cShuffle}_n \leq_W (\mathsf{IsFinite})^{2^n-1}$.

Sketch of the proof: the main idea is to use one instance p_C of IsFinite for every non-empty $C \subseteq n$. We scan all the intervals $I_j \subset \mathbb{Q}$, recording with $p_C(s) = 0$ if we need to change inteval at step s. Since for at least one C $\mathsf{IsFinite}(p_C) = 0$, a valid solution is a $\subseteq$-minimal such C.

Lemma

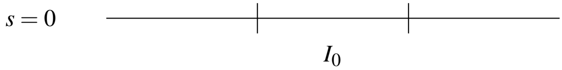
Let ECT_n *be the restriction of* ECT *to colorings with exactly* n *colors. Then,* $\mathsf{ECT}_n \leq_{\mathrm{W}} \mathsf{iShuffle}_n$.

Sketch of the proof: we use the fact that there is a computable order-preserving bijection between the dyadic numbers and $\mathbb{Q}$. Given $f : \mathbb{N} \to n$, we define $c_f : \mathbb{Q} \to n$ by $c\left(\frac{a}{2^h}\right) = f(h)$. Then, the diameter of the shuffle gives the bound b for f.

Lemma

$\mathsf{iShuffle}_n \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^{n-1}$

Case of $n = 2$:



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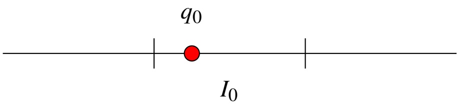
Lemma

iShuffle$_n \leq_W$ TC$_{\mathbb{N}}^{n-1}$

Case of $n=2$:

$s=0$



$e(0)=\{\langle 0,a\rangle : a\neq 0\}$

iShuffle

Lemma

Let ECT_n *be the restriction of* ECT *to colorings with exactly* n *colors. Then,* $\mathsf{ECT}_n \leq_{\mathrm{W}} \mathsf{iShuffle}_n$*.*

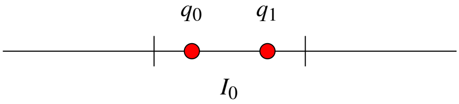
Sketch of the proof: we use the fact that there is a computable order-preserving bijection between the dyadic numbers and $\mathbb{Q}$. Given $f : \mathbb{N} \to n$, we define $c_f : \mathbb{Q} \to n$ by $c\left(\frac{a}{2^h}\right) = f(h)$. Then, the diameter of the shuffle gives the bound b for f.

Lemma

$\mathsf{iShuffle}_n \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^{n-1}$

Case of $n = 2$:

$s = 1$



$e(1) = \{\langle 1, a\rangle : a \neq 0\}$

iShuffle

Lemma

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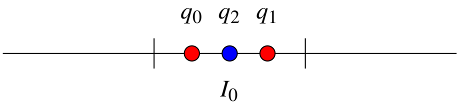
Sketch of the proof: we use the fact that there is a computable order-preserving bijection between the dyadic numbers and $\mathbb{Q}$. Given $f : \mathbb{N} \to n$, we define $c_f : \mathbb{Q} \to n$ by $c\left(\frac{a}{2^h}\right) = f(h)$. Then, the diameter of the shuffle gives the bound b for f.

Lemma

$\mathsf{iShuffle}_n \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^{n-1}$

Case of $n = 2$:

$s = 2$



$e(1) = \{\langle 1, a\rangle : a \neq 0\}$

iShuffle

Lemma

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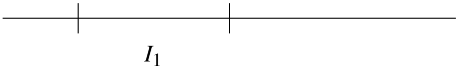
Case of $n = 2$:

$s = 2$



$e(2) = \{\langle 0,0\rangle, \langle 1,0\rangle\}$

iShuffle

Lemma

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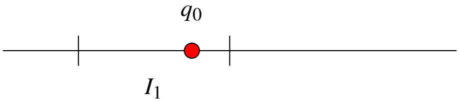
Sketch of the proof: we use the fact that there is a computable order-preserving bijection between the dyadic numbers and $\mathbb{Q}$. Given $f:\mathbb{N}\to n$, we define $c_f:\mathbb{Q}\to n$ by $c\left(\frac{a}{2^h}\right)=f(h)$. Then, the diameter of the shuffle gives the bound b for f.

Lemma

iShuffle$_n \leq_W$ TC$_{\mathbb{N}}^{n-1}$

Case of $n=2$:

$s=3$



$e(3)=\{\langle 2,a\rangle : a\neq 1\}$

iShuffle

Lemma

Let ECT_n *be the restriction of* ECT *to colorings with exactly* n *colors. Then,* $\mathsf{ECT}_n \leq_{\mathrm{W}} \mathsf{iShuffle}_n$*.*

Sketch of the proof: we use the fact that there is a computable order-preserving bijection between the dyadic numbers and $\mathbb{Q}$. Given $f : \mathbb{N} \to n$, we define $c_f : \mathbb{Q} \to n$ by $c\left(\frac{a}{2^h}\right) = f(h)$. Then, the diameter of the shuffle gives the bound b for f.

Lemma

$\mathsf{iShuffle}_n \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^{n-1}$

Case of $n = 2$:
Sketch of a sketch of the proof: we use the i^{th} instance of $\mathsf{TC}_{\mathbb{N}}$ to check if the inteval I_j contains $n-(i+1)$ many colors. A valid answer will be given by checking the longest chain of inclusion of intervals.

Together with known results, we thus get

$$\mathsf{ECT}_n \equiv_{\mathrm{W}} \mathsf{iShuffle}_n \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^{n-1} \text{ and hence } \mathsf{TC}_{\mathbb{N}}^{*} \equiv_{\mathrm{W}} \mathsf{iShuffle}$$

Shuffle: putting it all together

From the previous slides, we have

- $\bigsqcup_{n\in\mathbb{N}} \mathsf{IsFinite}^n \equiv_W \bigsqcup_{n\in\mathbb{N}} \mathsf{cShuffle}_n$, so $\mathsf{cShuffle} \equiv_W (\mathsf{LPO}')^*$
- $\bigsqcup_{n\in\mathbb{N}} \mathsf{ECT}_n \equiv_W \bigsqcup_{n\in\mathbb{N}} \mathsf{iShuffle}_n$, so $\mathsf{iShuffle} \equiv_W \mathsf{TC}_{\mathbb{N}}^*$

Since $\mathsf{cShuffle} \times \mathsf{iShuffle} \leq_W \mathsf{Shuffle}$, it immediately follows that $(\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^* \leq_W \mathsf{Shuffle}$.

Moreover, One can intertwine the proofs above to get that $\mathsf{Shuffle}_n \leq_W (\mathsf{LPO}' \times \mathsf{TC}_{\mathbb{N}})^{2^n-1}$.

This suffices to show that

$$\mathsf{Shuffle} \equiv_W (\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^*,$$

as we wanted.

The case of $\mathsf{ART}_{\mathbb{Q}}$

It is easy to see that Shuffle $\le_W \mathsf{ART}_{\mathbb{Q}}$, hence $(\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^* \le_W \mathsf{ART}_{\mathbb{Q}}$.

The other direction is more complicated: the rough idea is to go back to the case of groups via Green theory.

For a semigroup $(S, \cdot)$, define the **Green preorders** as follows:

- $s \le_{\mathscr{R}} t$ if and only if $s = t$ or $s \in tS = \{ta : a \in S\}$ (suffix order)
- $s \le_{\mathscr{L}} t$ if and only if $s = t$ or $s \in St = \{at : a \in S\}$ (prefix order)
- $s \le_{\mathscr{H}} t$ if and only if $s \le_{\mathscr{R}} t$ and $s \le_{\mathscr{L}} t$
- $s \le_{\mathscr{J}} t$ if and only if $s \le_{\mathscr{R}} t$ or $s \le_{\mathscr{L}} t$ or $s \in StS = \{atb : (a,b) \in S^2\}$ (infix order)

The associated equivalence relations are written $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{J}$.

- If $(S, \cdot)$ is a finite semigroup, $H \subseteq S$ an $\mathscr{H}$-class, and some $a, b \in H$ satisfy $a \cdot b \in H$ then, for some $e \in H$, $(H, \cdot, e)$ is a group.
- For $x, y \in S$, if $x \le_{\mathscr{R}} y$ and x, y $\mathscr{J}$-equivalent, then x, y are $\mathscr{R}$-equivalent.
- Same thing but with $\mathscr{L}$.

The case of $\mathsf{ART}_{\mathbb{Q}}$ (cont.)

Notice: given $c : [\mathbb{Q}]^2 \to S$, the map $c_{\mathscr{J}} : [\mathbb{Q}]^2 \to \mathscr{J}$-classes is ordered.

So, with an application of $\mathsf{ORT}_{\mathbb{Q}}$, we move to an interval $]u,v[$ where all elements are in the same $\mathscr{J}$-class.

We apply iShuffle to $c' :]u,v[\to S^2$, where $c'(x) = (c(u,x), c(x,v))$, finding $]a,b[$. We claim that it is a valid solution to $\mathsf{iART}_{\mathbb{Q}}$. Let $x < y \in]a,b[$ with $c'(x) = c'(y) = (l,r)$. Then

- $r = c(x,y) \cdot r$, so $r \leq_{\mathscr{R}} c(x,y)$; but $c(x,y) \mathrel{\mathscr{J}} r$ implies $c(x,y) \mathrel{\mathscr{R}} r$.
- $l = l \cdot c(x,y)$, so $l \leq_{\mathscr{R}} c(x,y)$; but $c(x,y) \mathrel{\mathscr{J}} r$ implies $c(x,y) \mathrel{\mathscr{L}} l$.

Hence, $c'(x) = c'(y)$ implies $c(x,y)$ in the same $\mathscr{H}$-class, and we can deal with them as in the case of groups.

So, $\mathsf{iART}_{\mathbb{Q}} \leq \mathsf{iShuffle} \star \mathsf{ORT}_{\mathbb{Q}}$.

Lemma

$\mathsf{ORT}_{\mathbb{Q}} \equiv_{\mathrm{W}} \mathsf{LPO}^* \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$. *Hence,* $\mathsf{iART}_{\mathbb{Q}} \leq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$. *Similarly for the others.*

More on colors

Notice that in the previous slide the argument was somewhat delicate: we needed that $\mathsf{iART}_{\mathbb{Q},n} \leq_{\mathrm{W}} \mathsf{iShuffle}_{n^2} \star \mathsf{ORT}_{\mathbb{Q}}$ in order to eliminate $\mathsf{ORT}_{\mathbb{Q}}$. Indeed, in general, it would not hold that $\mathsf{TC}^*_{\mathbb{N}} \star \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}} \times \mathsf{C}_{\mathbb{N}}$.

- A Weihrauch degree is a **fractal** if it contains $F :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ such that for every $w \in \mathbb{N}^*$, either $[w] \cap \operatorname{dom} F = \emptyset$, or $F\restriction_{[w]} \equiv_{\mathrm{W}} F$.
- It is a **closed fractal** if there is a **total** such F.

The elimination of $\star \mathsf{C}_{\mathbb{N}}$ was related to the fact that we were working with closed fractals: in general, this is why we choose to code instances the way we did. E.g. $\mathsf{Shuffle} \equiv_{\mathrm{W}} \bigsqcup_{n \in \mathbb{N}} \mathsf{Shuffle}_n$ is not even a fractal.

There would be another option: removing the set of colors from the input, and only promising that it is finite. This gives rise to fractals instead of closed fractals (e.g., $\mathsf{RT}^1_{\mathbb{N}}$ vs $\bigsqcup_{n \in \mathbb{N}} \mathsf{RT}^1_n$).

$\mathsf{ORT}_{\mathbb{N}}$ and $\mathsf{ART}_{\mathbb{N}}$

Spoiler alert: we have that $\mathsf{ORT}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{ART}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^{*} \times (\mathsf{LPO}')^{*}$.

The case for $\mathbb{N}$ is somewhat more complicated: if nothing else, by the way we formulated them, it is not even clear why the principles should be first-order.

$\mathsf{iORT}_{\mathbb{N}}$	Input: $((S,\cdot),c)$ such that $c:[\mathbb{N}]^2 \to S$ is right-ordered Output: n_0 such that there is H i.h. with $\lvert H \cap [0,n_0] \rvert \geq 1$
$\mathsf{iART}_{\mathbb{N}}$	Input: $((S,\cdot),c)$ such that $c:[\mathbb{N}]^2 \to S$ is additive Output: n_0 such that there is H i.h. with $\lvert H \cap [0,n_0] \rvert \geq 2$

$\mathsf{cORT}_{\mathbb{N}}$ and $\mathsf{cART}_{\mathbb{N}}$ are defined in the obvious way.

Easy to see that $\mathsf{ECT}_{n} \leq_{\mathrm{W}} \mathsf{iORT}_{\mathbb{N},2^n}$ by considering $(\mathscr{P}(n),\subseteq)$ (and $\mathsf{ECT}_{n} \leq_{\mathrm{W}} \mathsf{iART}_{\mathbb{N},2^n}$ by considering $(\mathscr{P}(n),\cup)$). Even easier to see that $\mathsf{IsFinite} \leq_{\mathrm{W}} \mathsf{cORT}_{\mathbb{N}}$ and $\mathsf{IsFinite} \leq_{\mathrm{W}} \mathsf{cART}_{\mathbb{N}}$.

$\mathsf{ORT}_{\mathbb{N}}$

We deal with $\mathsf{ORT}_{\mathbb{N}}$, the case of $\mathsf{ART}_{\mathbb{N}}$ being similar but more complicated (i.e. it requires some more Green theory magic).

Morally: an $\mathsf{ORT}_{\mathbb{N}}$-solution to $c : [\mathbb{N}]^2 \to P$ is given by a color $p \in P$ $\leq_P$ maximal such that an infinite c-homogeneous set for p exists and an n_1 such that $\forall n_1 < x < y$, $c(x,y) \not\geq_P p$.

If we have these two numers, we can build a solution recursively as follows: suppose H_n is given with $n_1 < \min H_n$, look for the smallest $m > \max H_n$ such that $c(H_n, m) = p$ and there is $m' > m$ with $c(m, m') = p$. Let $H_{n+1} = H_n \cup \{m\}$.

It can be shown that p and n_1 can be found using $(\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}^*$, and moreover that $\mathsf{cORT}_{\mathbb{N}} \leq_{\mathrm{W}} (\mathsf{LPO}')^*$ and $\mathsf{iORT}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{ECT}$.

Conclusions

Summary of results:

- $\mathsf{iART}_{\mathbb{Q}} \equiv_{\mathrm{W}} \mathsf{iShuffle} \equiv_{\mathrm{W}} \mathsf{iART}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{iORT}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$
- $\mathsf{cART}_{\mathbb{Q}} \equiv_{\mathrm{W}} \mathsf{cShuffle} \equiv_{\mathrm{W}} \mathsf{cART}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{cORT}_{\mathbb{N}} \equiv_{\mathrm{W}} (\mathsf{LPO}')^*$
- $\mathsf{ART}_{\mathbb{Q}} \equiv_{\mathrm{W}} \mathsf{Shuffle} \equiv_{\mathrm{W}} \mathsf{ART}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{ORT}_{\mathbb{N}} \equiv_{\mathrm{W}} (\mathsf{LPO}')^* \times \mathsf{TC}_{\mathbb{N}}$
- $\mathsf{ORT}_{\mathbb{Q}} \equiv_{\mathrm{W}} \mathsf{LPO}^*$

Moreover, we could have shown (see [PPS23]) that

- $(\eta \longrightarrow (\eta)^1_{<\infty}) \equiv_{\mathrm{W}} \mathsf{i}(\eta \longrightarrow (\eta)^1_{<\infty}) \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^*$
- $\mathsf{c}(\eta \longrightarrow (\eta)^1_{<\infty}) \equiv_{\mathrm{W}} \mathsf{RT}^1_+$

In a way, this work can be seen as a contribution to the study of the correspondence between combinatorial systems from reverse mathematics and Weihrauch degrees (see [BR17] for many other results in this direction).

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Thank you!