The Logical Firmament

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"And now for something completely different."
Monty Python

Epistemology of logic and math

How do we come to know logical and mathematical truths?

Historical focus exclusively on elementary logical truths:

> *But what about logical truth? Is it knowable a priori? And if so, how? In the case of some logical truths, the explanation for how we have come to know them will be clear: we will have deduced them from others. So our question concerns only the most elementary laws of sentential or first-order logic. (Boghossian 1997, p. 101)*

Questions like: How do we know *modus ponens* is valid? How do we know addition is associative?

Not sure how much sense these questions make. (Isn't it to some degree stipulative?)
But also, doesn't track the interesting part of practice among working logicians and mathematicians.

Logical non-omniscience

Does "$(P \supset \sim P) \supset \sim P$" express a logical truth?

I know all the elementary truths in the vicinity, but I don't know that. How do I go about learning it?

First step towards an answer: What precisely do I fail to know when I fail to know whether that sentence expresses a logical truth?

Narrowing ignorance

What is Peter Vranas's office number?

I know:

- Vranas's office is on the fifth floor of Helen C. White Hall.
- It's next to mine.
- My office number is 5169.
- The numbers increase in Vranas's direction.

I don't know:

- Whether the office numbers increase by ones or twos.

This is a fact I lack such that, if I knew it, I'd know Vranas's office number.

Outline

1. Catenary truths

2. A challenge

3. A possible solution

A fun fact

Moving a knight 46 times on a standard chessboard will never leave it adjacent to the square on which it began.

One helpful explanation of this fact includes the following:

- A standard chessboard has squares of two colors, and adjacent squares are opposite colors.
- Moving a knight takes it from a square of one color to the other.
- 46 is an even number.
- Toggle Fact: Given a system with two possible states, and an operation that toggles the system's state, even-numbered repetitions of the operation return the system to its original state.

A catenary truth says something about the result of applying a sequence of operations (or operation types).

- The sequence may contain multiple iterations of the same operation (as in the Toggle Fact), it may contain different operations, or it may contain multiple iterations of multiple operations.
- Catenary truths are abstract and *a priori*.
- Catenary truths occur in hierarchies of generality.
  Examples of catenary truths:
    - Knight Fact: 46 knight moves won't leave you on an adjacent square.
    - An even number of knight moves won't leave you on an adjacent square.
    - Toggle Fact: Given a system with two possible states, and an operation that toggles the system's state, even-numbered repetitions of the operation return the system to its original state.
- The Toggle Fact also helps explain such catenary truths as:
    - Multiplying a positive number by 46 negative numbers yields a positive result.
    - A proposition negated 46 times has the same truth-value as the original.

Does "$(P \supset {\sim}P) \supset {\sim}P$" express a logical truth?

I know:

- What truth-functional operations are represented by "$\supset$" and "$\sim$".
- What it means for a sentence to express a logical truth.

I don't know:

- Convoluted Operations Fact: Suppose we have a system with two possible states, one of which we'll call the "designated state". Whatever state the system is in initially, we take that state and toggle it. We then take the result and make it the second input into a binary function that yields the designated state unless its first input is the designated state and its second input is the non-designated state—with the first input being the initial state again. We then take the output of *that*, and make it the first input of the same binary function, with the second input being the initial state toggled. Regardless of the initial state of the system, the result of all these operations will be that the final application of the binary function yields the designated state.

Mathematical non-omniscience

What is $295 + 437$?

I know:

- What operations I'd have to carry out in what order to answer this question.
- Um. . . that the answer ends in a 2?

I don't know:

- The full result of carrying those operations out.

Catenary truths are the interesting part of logic, math, chess. . . .

Outline

Anything to see here?

> *But what about logical truth? Is it knowable a priori? And if so, how? In the case of some logical truths, the explanation for how we have come to know them will be clear: we will have deduced them from others. So our question concerns only the most elementary laws of sentential or first-order logic. (Boghossian 1997, p. 101)*

Two interesting metaphysical facts about catenary truths:

- They are independent of us, not "up to us".
- They are distinct from the elementary truths characterizing the operations they involve.

So how does deduction from the elementary truths get us this kind of knowledge?

A proposal

Gödel's Platonism and epistemology of intuition:

> *The objects of transfinite set theory... clearly do not belong to the physical world and even their indirect connection with physical experience is very loose...*
>
> *But, despite their remoteness from sense experience, we do have a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception. (revised version of "What is Cantor's Continuum Problem?", 1964)*

Benacerraf's challenge

For Hermione to know that the black object she is holding is a truffle is for her (or at least requires her) to be in a certain (perhaps psychological) state. It also requires the cooperation of the rest of the world, at least to the extent of permitting the object she is holding to be a truffle. Further—and this is the part I would emphasize—in the normal case, that the black object she is holding is a truffle must figure in a suitable way in a causal explanation of her belief that the black object she is holding is a truffle. . . .

I find [Gödel's] picture both encouraging and troubling. What troubles me is that without an account of how the axioms "force themselves upon us as being true," the analogy with sense perception and physical science is without much content. For what is missing is. . . an account of the link between our cognitive faculties and the objects known. (1973)

Outline

Benacerraf-style questions

- In what relation does reasoning place us with respect to complex/catenary truths?
- How does it do that?

Beginning of an answer: I don't know whether "$(P \supset \sim P) \supset \sim P$" expresses a logical truth because I lack a specific piece of information, captured by the Convoluted Operations Fact.

So what do I *do*?
Well, I can make a truth-table, either on paper or in my head. I make it in the typical way, with "$\sim$" switching truth-values and "$\supset$" yielding "T" unless its antecedent is true and its consequent false.
When I'm done, the truth-table has a "T" on each row under the sentence's main connective.
So I form a belief that the sentence expresses a logical truth.

Why does the truth-table come out that way?

Convoluted Operations Fact: Suppose we have a system with two possible states, one of which we'll call the "designated state". Whatever state the system is in initially, we take that state and toggle it. We then take the result and make it the second input into a binary function that yields the designated state unless its first input is the designated state and its second input is the non-designated state—with the first input being the initial state again. We then take the output of *that*, and make it the first input of the same binary function, with the second input being the initial state toggled. Regardless of the initial state of the system, the result of all these operations will be that the final application of the binary function yields the designated state.

I am looking for a particular piece of information. I carry out a calculation, and use it to form a belief. That calculation is an instance of the very piece of information I was looking for.

Put another way

Wittgenstein: "What is the difference between a calculation and an experiment?"

I don't know what the laws of physics say happens when you smash these two kinds of particles together.
One way to find out: Smash a couple of those particles together, see what results.

I don't know what happens when certain types of operations are carried out in a particular sequence.
One way to find out: Carry out such operations in that sequence, see what results.

Applied to formal proofs

Suppose we are wondering if a particular sentence can be derived in a natural deduction system.
Seeking a catenary truth about whether the operations allowable in that system can be combined in order to yield a result.

By constructing the derivation, we answer that question. What we construct is an instance of the catenary truth.

<u>Big picture idea</u>: At the right level of generality, catenary truths apply both to the entities we investigate in math and logic, and to the processes we employ in that investigation.

Remaining questions/concerns

- How to apply the view to non-formal proofs.
- You still haven't said how we learn the atomic facts!
- What happens when reasoning goes wrong?
- Is logical knowledge still *a priori*?
- Is logical knowledge synthetic?
- How do we know the results of our reasoning processes are necessary?

Thank you!

Logical positivism

They simply record our determination to use words in a certain fashion. We cannot deny them without infringing the conventions which are presupposed by our very denial, and so falling into self-contradiction. And this is the sole ground of their necessity. (Ayer, Language, Truth, and Logic)

Dummett responds:

This account... leaves unexplained the status of the assertion that certain conventions have certain consequences. It appears that if we adopt the conventions registered by the axioms, together with those registered by the principles of inference, then we must adhere to the way of talking embodied in the theorem; and this necessity must be one imposed upon us, one that we meet with.