

Pointwise definable and Leibnizian extensions of models of arithmetic and set theory

Joel David Hamkins
O'Hara Professor of Philosophy and Mathematics
University of Notre Dame

Associate Faculty, Professor of Logic
University of Oxford

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Joint work

This talk includes joint work with W. Hugh Woodin, Kameryn Williams, Victoria Gitman, as well as solo work.

Pointwise definability

Definition

A model is *pointwise definable*, if every individual is definable without parameters.

The Math Tea argument

There must be some real numbers we can neither describe nor define, because there are uncountably many reals, but only countably many definitions.

Pointwise definable models may pose a problem...

Meanwhile:

“I can describe any number. Let me show you: you tell me a number, and I’ll tell you a description of it.”

–Horatio, age 8

Leibnizian models

Definition

A model M is *Leibnizian*, if distinct elements have different properties.

If $a \neq b$, then for some property φ ,

$$M \models \varphi[a] \quad \text{but} \quad M \models \neg\varphi[b].$$

Leibnizian models are thus precisely those that fulfill:

Leibniz principle on Identity of Indiscernibles

Indiscernible individuals are identical.

Goal Theorems

I aim to provide a flexible new proof of:

Goal Theorem 1

Every countable model of PA has a pointwise definable end-extension.

The same method applies in set theory.

Goal Theorem 2

Every countable model of ZF has a pointwise definable end-extension. Can achieve $V = L$ in the extension, or any other theory, if true in an inner model of $V = \text{HOD}$.

Goal Theorems

Goal Theorem 3

Every model of PA of size at most continuum has a Leibnizian extension.

The same method applies in set theory.

Goal Theorem 4

Every model of ZF of size at most continuum has a Leibnizian extension to a model of $V = L$, or indeed of any theory true in some inner model of $V = \text{HOD}$.

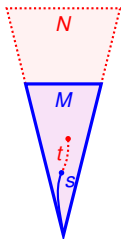
The proofs are both flexible and soft.

Universal algorithm

The method begins with a remarkable theorem of Woodin [Woo11].

Namely, there is a Turing machine program e with an amazing universal extension property:

- 1 It enumerates a finite sequence, and PA proves this.
- 2 In any model $M \models \text{PA}$, if the sequence is s , then for any desired t , there is an end-extension in which e computes t .

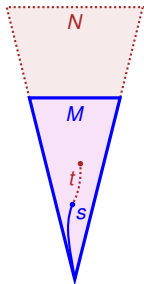


History: Woodin [Woo11], Blanck and Enayat [BE17; Bla17], simplified proof in [Ham18; Ham17].

Proof proceeds by a highly self-referential algorithm, “the petulant child.”

Generalization to Σ_m -elementary extensions

The result generalizes ([Ham18]) to provide a Σ_{m+1} -definable finite sequence, with the universal extension property with respect to Σ_m -elementary end-extensions $M \prec_{\Sigma_m} N$.



Again every model $M \models \text{PA}$ can realize any desired extension t in an end-extension N .

But the difference now is that Σ_m truth is preserved between M and N .

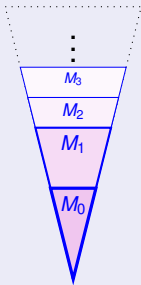
Pointwise definable end-extensions

Main theorem 1 (Hamkins)

Every countable model of PA has a pointwise definable end extension satisfying PA.

Proof.

Build a tower of progressively elementary extensions



$$M_0 \subseteq M_1 \prec_{\Sigma_1} M_2 \prec_{\Sigma_2} M_3 \prec_{\Sigma_3} \dots$$

Put a_0 last on the Σ_1 -definable sequence.

Then a_1 last on Σ_2 -sequence, and so on.

Limit model N is a model of PA.

Can arrange that every element becomes definable. So N is pointwise definable.



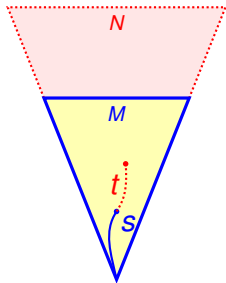
Universal definition in set theory

Woodin and I proved a set-theoretic analogue of the universal algorithm [HW17].

There is a Σ_2 definable finite sequence

$$a_0, a_1, \dots, a_n$$

with the universal extension property for top-extensions.



If sequence is s in countable $M \models \text{ZFC}$, then for any desired t , there is a top-extension $N \models \text{ZFC}$ in which the sequence is t .

The definition (complex, sophisticated) essentially looks for stages V_α that have no end-extension adding a next point a , even in any forcing extension, and when found, adds a anyway. “petulant child”

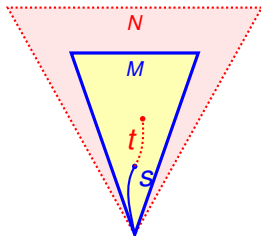
Σ_1 -definable universal sequence

Kameryn Williams and I proved [HW21] the Σ_1 -analogue.

There is a Σ_1 definable sequence

$$a_0, a_1, \dots, a_n$$

with the universal extension property for end-extensions.



If sequence is s in countable $M \models \text{ZFC}$, then for any desired t , there is an end-extension $N \models \text{ZFC}$ in which the sequence is t .

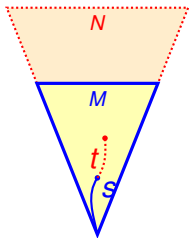
In fact, can get $N \models \overline{\text{ZFC}}$ for any theory true in some inner model W of M .

Generalization to Σ_m elementary end-extensions

In new work, I have been able to generalize to find a Σ_{m+1} definable sequence of ordinals

$$\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n$$

with the universal extension property for Σ_m -elementary end-extensions.



Every countable model $M \models \text{ZF}$ with sequence s has a Σ_m -elementary end-extension $N \models \text{ZF}$ realizing any desired extension t .

If $V = \text{HOD}$, can translate this to all objects, not just ordinals.

Pointwise definable extensions in set theory

Main theorem 2 (Hamkins)

Every countable model of ZF has a pointwise definable end extension. Indeed, it has such an extension satisfying $ZFC + V = L$.

Somewhat more general version:

Theorem (Hamkins)

Every countable model of ZF with an inner model of a c.e. theory \overline{ZFC} that includes $V = \text{HOD}$ has a pointwise definable end-extension satisfying \overline{ZFC} .

This realizes a certain *resurrection* property: whatever is true in some inner model can become true again in an end-extension, even a pointwise definable end-extension.

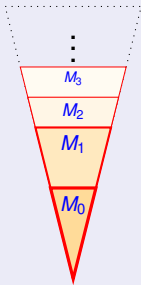
Pointwise definable extensions

Theorem (Hamkins)

Every countable model of $ZFC + V = HOD$ has a Σ_m -elementary pointwise definable end-extension.

Proof.

Build a tower of progressively elementary extensions



$$M_0 \subseteq M_1 \prec_{\Sigma_1} M_2 \prec_{\Sigma_2} M_3 \prec_{\Sigma_3} \dots$$

Put a_0 last on the Σ_1 -definable sequence.

Then a_1 last on Σ_2 -sequence, and so on.

Limit model N is a model of ZFC.

Can arrange that every element becomes definable. So N is pointwise definable.

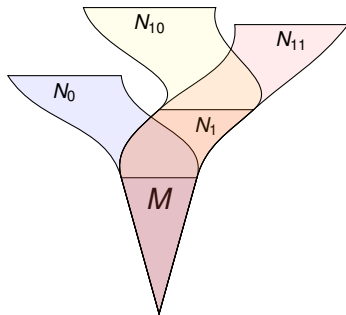


Consequences

Rich collection of consequences for the universal finite sequence

- Any object can become definable
- Pointwise definability comes by iterating this
- Pointwise definability is a switch
- No maximal Σ_m theory
- Modal logic of end-extension potentialism is exactly S4

The tree of top-extensions



Radical-branching potentialism.

Leibnizian analogue

Goal

To prove the analogues for Leibnizian extensions in place of pointwise definability.

- 1 Pointwise definable models (in a finite language) must be countable.
- 2 Leibnizian models have size at most continuum.

Question

Does every model of arithmetic (and set theory) of size continuum have a Leibnizian extension?

Yes.

Leibnizian extensions for PA

Theorem

Every model of PA of size at most the continuum admits a Leibnizian extension. Indeed, for any particular natural number m , the model admits a Σ_m -elementary Leibnizian extension.

Proof strategy. Given $M_0 \models \text{PA}$ of size at most continuum, construct a progressively elementary tower

$$M_0 \prec M_1 \prec_{\Sigma_m} M_2 \prec M_3 \prec_{\Sigma_{m+1}} M_4 \prec M_5 \prec_{\Sigma_{m+2}} \cdots$$

- Even stages, fully elementary. Create a countable set of points from which previous elements are discernible.
- Odd stages, progressively elementary. Make those points definable.

Even stages

At even stages, we create discernibility relative to countably many new constants.

Suppose M has size continuum. Assign to each $a \in M$ a binary sequence $s_a \in 2^\omega$.

Consider the theory

$T = \Delta(M) + "c_n \text{ codes a set with } a \text{ as member}"$, when $s_a(n) = 1$.

Finitely consistent, hence consistent.

So we find $M \prec N$ with countably many new elements c_n that discern the elements of M .

Odd stages

At odd stages, we make the accumulating constants definable.

$$M_0 \prec M_1 \prec_{\Sigma_m} M_2 \prec M_3 \prec_{\Sigma_{m+1}} M_4 \prec M_5 \prec_{\Sigma_{m+2}} \cdots$$

Thus, we build a progressively elementary tower, in which the constants all eventually become definable.

These constants support the discernibility of the other elements.

And so the limit model is Leibnizian, as desired. \square

Thank you.

Slides and articles available on <http://jdh.hamkins.org>.

Joel David Hamkins

O'Hara Professor Philosophy and Mathematics

University of Notre Dame

Associate Faculty Member, Professor of Logic

University of Oxford

References

- [BE17] Rasmus Blanck and Ali Enayat. “Marginalia on a theorem of Woodin”. *J. Symb. Log.* 82.1 (2017), pp. 359–374. ISSN: 0022-4812. DOI: 10.1017/jsl.2016.8.
- [Bla17] Rasmus Blanck. “Contributions to the Metamathematics of Arithmetic. Fixed points, Independence, and Flexibility”. PhD thesis. University of Gothenburg, 2017. ISBN: 978-91-7346-917-3.
<http://hdl.handle.net/2077/52271>.
- [Ham17] Joel David Hamkins. *The universal algorithm: a new simple proof of Woodin’s theorem*. Mathematics and Philosophy of the Infinite. 2017. <http://jdh.hamkins.org/the-universal-algorithm-a-new-simple-proof-of-woodins-theorem/> (version 28 May 2017).
- [Ham18] Joel David Hamkins. “The modal logic of arithmetic potentialism and the universal algorithm”. *Mathematics arXiv* (2018). Under review, pp. 1–35. arXiv:1801.04599[math.LO].
<http://wp.me/p5M0LV-1Dh>.

- [HW17] Joel David Hamkins and W. Hugh Woodin. “The universal finite set”. *Mathematics arXiv* (2017). Manuscript under review, pp. 1–16. arXiv:1711.07952 [math.LO]. <http://jdh.hamkins.org/the-universal-finite-set>.
- [HW21] Joel David Hamkins and Kameryn J. Williams. “The Σ_1 -definable universal finite sequence”. *Journal of Symbolic Logic* (2021). DOI: 10.1017/jsl.2020.59. arXiv:1909.09100 [math.LO].
- [Woo11] W. Hugh Woodin. “A potential subtlety concerning the distinction between determinism and nondeterminism”. In: *Infinity*. Cambridge University Press, 2011, pp. 119–129.