Observations and questions on the structure of the Weihrauch degrees

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2017: The survey

Vasco Brattka, Guido Gherardi & Arno Pauly:
Weihrauch Complexity in Computable Analysis.
arXiv 1707.03202

And an update

What happened since? What are some interesting open questions?

Arno Pauly:
An update on Weihrauch complexity, and some open questions.
arXiv 2008.11168

A very short overview

- Weihrauch reducibility compares multivalued functions between represented spaces.
- The induced degrees have a rich algebraic structure.
- Many mathematical theorems can be interpreted as multivalued functions, with the associated Weihrauch degrees measuring the computational content of the theorem.
- The algebraic operations have logic-like meanings regarding such theorems.
- Many concrete theorems have been classified via Weihrauch reducibility; and this classification is reminiscent of reverse mathematics and Brouwerian counterexamples.
- Various techniques have been developed to prove separation results.

Represented spaces and computability

Definition

A *represented space* $\mathbf{X}$ is a pair (X, δ_X) where X is a set and $\delta_X :\subseteq \mathbf{2}^{\mathbb{N}} \to X$ a surjective partial function.

Definition

$F :\subseteq \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ is a realizer of $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, iff $\delta_Y(F(p)) \in f(\delta_X(p))$ for all $p \in \mathrm{dom}(f\delta_X)$.

$$\begin{array}{ccc} \mathbf{2}^{\mathbb{N}} & \xrightarrow{F} & \mathbf{2}^{\mathbb{N}} \\ \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_Y} \\ \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \end{array}$$

Definition

$f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ is called computable (continuous), iff it has a computable (continuous) realizer.

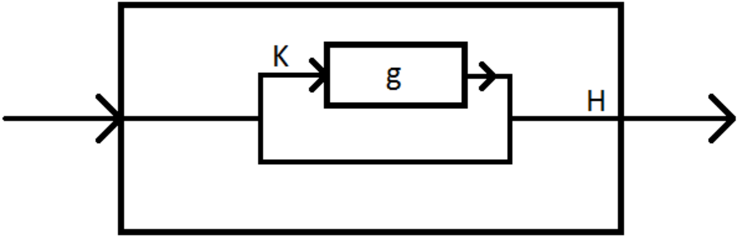
Weihrauch-reducibility

Definition

For $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $g :\subseteq \mathbf{V} \rightrightarrows \mathbf{W}$ say

$$f \leq_W g$$

iff there are computable $H, K :\subseteq \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$, such that $H\langle \mathrm{id}_{\mathbb{N}^{\mathbb{N}}}, GK\rangle$ is a realizer of f for every realizer G of g. $\mathfrak{W}$ denotes the Weihrauch degrees.



Weihrauch reducibility on Baire space

Proposition

For $f, g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ *we have that* $f \leq_W g$ *iff there are computable* $H, K \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *with* $K : \mathrm{dom}(f) \to \mathrm{dom}(g)$ *such that* $H(\langle p, q \rangle) \in f(p)$ *for all* $q \in g(K(p))$*.*

What people are working on

- Most work on Weihrauch degrees is about classifying specific theorems.
- Then there is work on creating a “scaffolding” of stuff like closed choice principles.
- But only a few papers on the structure of the Weihrauch degrees.
- See http://cca-net.de/publications/weibib.php

Outline

Distributive lattice

Theorem (Brattka & Gherardi; Pauly)

The Weihrauch degrees form a distributive lattice;

- *with join* $\sqcup$ *induced by* $(f \sqcup g) :\subseteq \mathbf{X} + \mathbf{U} \rightrightarrows \mathbf{Y} + \mathbf{U}$, $(f \sqcup g)(0, x) = (0, f(x))$ *and* $(f \sqcup g)(1, y) = (1, g(y))$,
- *and with meet* $\sqcap$ *induced by* $(f \sqcap g) :\subseteq \mathbf{X} \times \mathbf{U} \rightrightarrows \mathbf{Y} + \mathbf{V}$, $(f \sqcap g)(x, y) = (0 \times f(x)) \cup (1 \times g(y))$.

Special degrees

- The least element is 0, the trivially true principle without instances.
- With 1 we denote the degree of $\mathrm{id}_{\mathbb{N}^\mathbb{N}}$ comprised of all computable problems with a computable instance.
- And $\emptyset$ is the top element (which is probably fake).

Incompleteness

Theorem (Higuchi & Pauly)

No non-trivial suprema exist in the Weihrauch lattice, meaning either $\sqcup_{i\in\mathbb{N}} f_i$ does not exist, or there is some $N \in \mathbb{N}$ with $\sqcup_{i\in\mathbb{N}} f_i = \sqcup_{i\leq N} f_i$.

Theorem (Higuchi & Pauly)

Some non-trivial infima exist, others do not.

Corollary

$\mathfrak{W}$ *and* $\mathfrak{W}^{\mathrm{op}}$ *are not isomorphic.*

Heyting algebra?

Question (Brattka & Gherardi)

Is the Weihrauch lattice a Brouwer algebra, i.e. does

$$\inf_{\leq_W}\{h \mid g \leq_W f \sqcup h\}$$

exist for all f, g?

Theorem (Higuchi & Pauly)

The Weihrauch lattice is neither a Brouwer not a Heyting algebra.

Medvedev degrees

Definition (Medvedev reducibility)

For $A, B \subseteq \mathbb{N}^{\mathbb{N}}$, $A \leq_M B$ iff $\exists F : B \to A$, F computable. Let $\mathfrak{M}$ denote the Medvedev degrees.

Theorem (Brattka & Gherardi)

$A \mapsto c_A$, where $c_A(p) = A$, is a meet-semilattice embedding of $\mathfrak{M}$ into $\mathfrak{W}$.

Theorem (Higuchi & Pauly)

$A \mapsto d_A$, where $d_A : A \to \{0\}$, is a lattice embedding of $\mathfrak{M}^{\mathrm{op}}$ into $\mathfrak{W}$. In fact, it is an isomorphism between $\mathfrak{M}^{\mathrm{op}}$ and $\{f \in \mathfrak{W} \mid f \leq_W 1\}$.

Question

Is there a lattice-embedding of $\mathfrak{M}$ into $\mathfrak{W}$?

Many-one degrees

Definition (Many-one reductions)

For $A, B \subseteq \mathbb{N}$, let $A \leq_m B$ iff there is a computable $F : \mathbb{N} \to \mathbb{N}$ with $F^{-1}(B) = A$.

Theorem (Brattka & Pauly)

The many-one degrees embed into $\mathfrak{W}$.

Proof.

Let $p, q \in \mathbb{N}^\mathbb{N}$ be Turing incompatible. Map $A \subseteq \mathbb{N}$ to $\chi_A^{p,q} : \mathbb{N} \to \{p, q\}$ where $(\chi_A^{p,q})^{-1}(p) = A$. $\square$

Continuous Weihrauch reducibility

Definition

We say that f is continuously Weihrauch reducible to g (notation $f \leq_{\mathrm{W}}^{*} g$) if f is Weihrauch reducible to g relative to some oracle.

Same thing as replacing “computable” by “continuous” in the previous definition.

Observation ((exaggerated))

For the study of “real-life” problems, there is no difference between $\leq_W$ and $\leq_W^*$ because reasonable reductions are effective, and reasonable separation arguments relativize.

Question

But what about structure? (Beyond the algebraic part)

Step functions

Definition

For $\mathbf{X} = \mathbf{2}^{\mathbb{N}}$ or $\mathbf{X} = [0, 1]$, and $x \in \mathbf{X}$, define

$$s_x : \mathbf{X} \to \{0, 1\}$$

by $s_x(y) = 0$ iff $y < x$ and $s_x(y) = 1$ for $y \geq x$.

From now on, I ignore the case where $x = w0^\omega \in \mathbf{2}^{\mathbb{N}}$.

Question (Westrick, Dagstuhl 2019)

What is the structure of the Weihrauch degrees of step functions s_x?

An initial result

Theorem (Kihara & Westrick; P. & Westrick)

The following are equivalent for $x, y \in \mathbf{X}$

1. $s_x \leq_W s_y$
2. $s_x \leq_W \hat{s}_B$
3. *There is a nondecreasing computable* $H : \mathbf{X} \to \mathbf{X}$ *such that* $H(x) = y$ *and* $H((0, x)) < y$

What functions are we looking at?

Proposition (P. & Westrick)

If $s_A \leq_W s_B$ *for B non-computable witnessed by H, then* $H^{-1}(B) = \{A\}$*. If H is injective on a neighborhood of A, then* $s_A \equiv_W s_B$*.*

So “interesting” reduction witnesses are injective at the critical point, but are not injective on any neighborhood of it.

Contrasting spaces

Proposition (P. & Westrick)

If we identify $\mathbf{2}^{\mathbb{N}}$ *with the Cantor middle third set in* $[0,1]$*, the degrees of step functions are preserved.*

Proposition (P. & Westrick)

If $A \in [0,1]$ *is generic and* $B \in \mathbf{2}^{\mathbb{N}}$ *is not computable, then*

$$(s_A : [0,1] \to \{0,1\}) \not\leq_W (s_B : \mathbf{2}^{\mathbb{N}} \to \{0,1\})$$

Some simply observations

Proposition (P. & Westrick)

If $s_A \leq_W s_B$ and B is non-computable, then $A \equiv_{\mathrm{T}} s_B$.

Proposition (P. & Westrick)

If A is hyper-immune free, B is non-computable and $s_A \leq_W s_B$, then $s_A \equiv_W s_B$.

Proposition (P. & Westrick)

If A is random or generic, then there are $A_0, A_1, \ldots$ all tt-equivalent to A such that $(s_{A_n})_{n\in\mathbb{N}}$ is an antichain in the Weihrauch degrees.

What really is “and”?

Definition

We call f *join-irreducible*, if $f \leq_{\mathrm{W}} g \sqcup h$ implies that $f \leq_{\mathrm{W}} g$ or $f \leq_{\mathrm{W}} h$.

Most “natural” Weihrauch degrees are join-irreducible.

Definition

Let $f \times g : \mathbf{X} \times \mathbf{U} \rightrightarrows \mathbf{Y} \times \mathbf{V}$ be defined via $(y, v) \in (f \times g)(x, u)$ iff $y \in f(x)$ and $v \in g(v)$.

Proposition (Brattka)

$(\mathfrak{W}, 0, 1, \sqcup, \times, ^*)$ *is a Kleene-algebra.*

Sequential composition

Definition

Let $f \star g = \sup_{\leq_W}\{F \circ G \mid F \leq_W f \wedge G \leq_W g\}$.

Theorem (Brattka & Pauly)

$\star$ actually is a total operation on Weihrauch degrees.

Theorem (Dzhafarov, Goh, Hirschfeldt, Patey & Pauly)

$\mathrm{RT}^2_2 \leq_W \mathrm{SRT}^2_2 \star \mathrm{COH}$, *but* RT^2_2 *and* $\mathrm{SRT}^2_2 \times \mathrm{COH}$ *are incomparable.*

Substructural logics

Theorem (Brattka & Pauly)

The minimum $\min_{\leq_W}\{h \mid f \leq_W g \star h\}$ *always exists (and is denoted by* $g \to f$*, but in general none of the following have to exist:*

1. $\inf_{\leq_W}\{h \mid f \leq_W h \star g\}$
2. $\inf_{\leq_W}\{h \mid f \leq_W g \times h\}$

This means that the Weihrauch degrees are not a model of any of the usual substructural logics people have studied.

Closure under composition

Definition (Neumann & Pauly)

An input for $f^\diamond$ is a description of an abstract register machine operating on represented spaces with computable functions and f as operations, together with an input on which the register machine halts. The output is whatever the register machine outputs.

This is *supposed* to capture closure under composition.

Characterizations

Proposition

f^* is the least Weihrauch degree above f satisfying $1 \leq_W f^*$ and $f^* \times f^* \equiv_W f^*$.

Theorem (Westrick 2020)

$f^\diamond$ is the least Weihrauch degree above f satisfying $1 \leq_W f^\diamond$ and $f^\diamond \star f^\diamond \equiv_W f^\diamond$.

- Open since CCA 2015
- There is a constant function f and a multivalued function g such that $f \leq_W g^\diamond$, but no fixed finite number of applications of g suffices

Algebraic structure, summary

We have the following operations on Weihrauch degrees:

1. $f \sqcap g$, returning either an answer to f or an answer to g (OR)
2. $f \sqcup g$, letting us choose between f and g (AND)
3. $f \times g$, letting us both f and g in parallel (AND)
4. $f \star g$, letting us first use g, then f (AND)
5. $f \rightarrow g = \min\{h \mid g \leq_W f \star h\}$ (Implication)
6. $f^*, f^\diamond$ letting us use f finitely many times, in parallel or consecutively (bang, bang)
7. $\widehat{f}$, letting us use f countably many times in parallel (bang)
8. (and more)

More definability?

- Clearly $\sqcup$, $\sqcap$, $\emptyset$, 0 are definable just by $\leq_{\mathrm{W}}$
- Are $\times$ or 1 definable from other operations? What about $\widehat{\ }$?

On the theory of Weihrauch degrees

- The Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into the Weihrauch degrees (via the Medvedev degrees).
- Thus, any universally quantified statement using $\sqcup$ and $\sqcap$ is either provable from the axioms of distributive lattices or false in $\mathfrak{W}$.
- Can we extend this to additional operations?
- A list of known axioms and non-axioms is available in “On the algebraic structure of Weihrauch degrees”, LMCS 2018

An example

Question

What can we say about $(\mathfrak{W}, \leq_W, \times)$*?*

- $(\mathfrak{W}, \leq_{\mathrm{W}})$ is a distributive lattice, $(\mathfrak{W}, \times)$ a commutative monoid.
- $\times$ is $\leq_{\mathrm{W}}$-monotone in both arguments. $\times$ distributes over $\sqcup$ (but not over $\sqcap$)
- But there is more, e.g. $x \sqcap (y \times z) \leq_{\mathrm{W}} x \times y$

Towards an answer

Definition (Neumann, Pauly, Pradic)

Fix $k \in \mathbb{N}$. We define a structure $(P_k, \sqsubseteq, \sqcap, \times)$ as follows:

1. The elements of P_k are finite subsets A of finite subsets of $\mathbb{N}$ together with some colouring $c : \bigcup\bigcup A \to k$.
2. We have $(A, c) \sqsubseteq (B, d)$ if there is some $\lambda : \bigcup\bigcup B \to \bigcup\bigcup A$ with $c \circ \lambda = d$ such that for any $y \in B$ there is some $x \in A$ with $x \subseteq \{\lambda(n) \mid n \in y\}$.
3. To define $(A, c) \sqcap (B, d)$, assume that $\bigcup\bigcup A \cap \bigcup\bigcup B = \emptyset$ and then set $(A \sqcup B) = A \cup B$.
4. To define $(A, c) \times (B, d)$, assume that $\bigcup\bigcup A \cap \bigcup\bigcup B = \emptyset$ and then set $(A \times B) = \{x \cup y \mid x \in A \wedge y \in B\}$.

Towards an answer II

Theorem (Neumann, Pauly, Pradic)

$(P_k, \sqsubseteq, \sqcap, \times)$ embeds into $(\mathfrak{W}, \leq_W, \sqcap, \times)$.

Conjecture

The positive fragment of the Π_1-theory of $(\mathfrak{W}, \leq_W, \sqcap, \times)$ is the intersection of the positive fragments of the Π_1-theories of $(P_k, \sqsubseteq, \sqcap, \times)$ over all $k \in \mathbb{N}$.