On finite state dimension and its point to set principle

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Where do I come from, where am I going to?

- Effectivize a notion so that it is useful in a computably defined world
- Effectivize it some more so that you can use it in a finite automata defined world
- Use the effective notion to prove results in the classical world
- Use the finite-state notion to prove results in the classical world

Hausdorff dimension in Cantor space

Definition

A *martingale* is $d : 2^{<\omega} \to [0,\infty)$ with $2d(\sigma) = d(\sigma 0) + d(\sigma 1)$

Definition

For $s \in [0,1]$, and $x \in 2^\omega$, a martingale d *s-succeeds on x* if

$$\limsup_n d(x[1..n])2^{(s-1)n} = \infty$$

A martingale d *s-succeeds on $A \subseteq 2^\omega$* if d s-succeeds on every $x \in A$

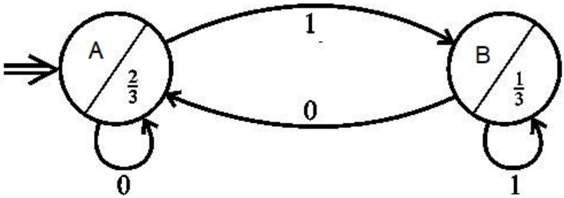
Theorem (Lutz 2003)

The Hausdorff dimension of $A \subseteq 2^\omega$, is

$$\dim_{\mathrm{H}}(A) = \inf \{ s \,|\exists d \text{ that } s\text{--succeeds on } A \}$$

Finite-state effectivization

- Restrict to martingales that can be computed by a finite-state automata
- We lose the identification of Cantor and Euclidean space, **the alphabet is important**
- Let Σ be a finite alphabet
- We consider finite-state gamblers with input alphabet Σ



Finite-state dimension

Definition (Dai et al 2004)

The *finite-state dimension* of $A \subseteq \Sigma^\omega$, is

$$\dim_{\mathrm{FS}}(A) = \inf \{ s \,|\, \exists d \text{ that } s\text{--succeeds on } A \}$$

Finite-state gamblers: strong dichotomy theorem

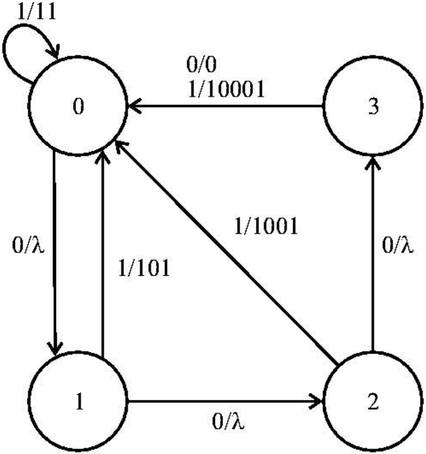
Theorem (Schnorr Stimm 72)

For each finite-state computable martingale d and each $x \in \Sigma^\omega$, one of the following holds

1. *There is an s for which d s-succeeds on x*
2. *$d(x[1..n])$ is constant for all sufficiently large n*
3. *There is an $\alpha < 1$ such that $d(x[1..n]) < \alpha^n$ for all sufficiently large n*

Information Lossless Finite-state compressors (ILFSC)

- We consider finite-state transducers with input and output alphabet Σ
- Information Lossless: The input can be recovered from the output and the end state



Information Lossless Finite-state compressors (ILFSC)

For each ILFSC M, $x \in \Sigma^\omega$,

$$\rho_M(x) = \liminf_n \frac{|M(x[1..n])|}{n}$$

Theorem (Lempel Ziv 78)

There is an IL compressor, LZ, that can be computed in time $n \log^c n$ such that LZ compresses asymptotically as much as any ILFSC, that is, for any M ILFSC and any $x \in \Sigma^\omega$

$$\rho_{LZ}(x) \le \rho_M(x)$$

IL Finite-state compressors characterize FS dimension

For each ILFSC M, $x \in \Sigma^\omega$,

$$\rho_M(x) = \liminf_n \frac{|M(x[1..n])|}{n}$$

Theorem (Dai et al 2004)

Let $x \in \Sigma^\omega$, $A \subseteq \Sigma^\omega$

- $\dim_{\mathrm{FS}}(x) = \inf_{M \text{ ILFSC}} \rho_M(x)$
- $\dim_{\mathrm{FS}}(A) = \inf_{M \text{ ILFSC}} \sup_{x\in A} \rho_M(x)$

Can we do it Kolmogorov-complexity style?

Let T be a finite state transducer with input and output alphabet Σ,

$$\mathrm{K}^{T}(\sigma) = \min\{|p| \mid T(p) = \sigma\}$$

Theorem (Doty Moser 2006, revisited Kozachinskiy Shen 2021)

Let $x \in \Sigma^\omega$, $A \subseteq \Sigma^\omega$

- $\dim_{\mathrm{FS}}(x) = \inf_{T\ \mathrm{FST}} \liminf_n \frac{\mathrm{K}^T(x[1..n])}{n}$
- $\dim_{\mathrm{FS}}(A) = \inf_{T\ \mathrm{FST}} \sup_{x \in A} \liminf_n \frac{\mathrm{K}^T(x[1..n])}{n}$

More characterizations

- (Hitchcock 2003) In terms of log-loss of finite-state predictors
- (Bourke et al 2005, Kozachinskiy Shen 2021) In terms of (overlapping and non-overlapping) block-entropy
- (Lutz et al 2022) Weyl criterion for $\dim_{\mathrm{FS}}$ in $(0, 1)$

Borel Normality

$\Sigma_b = \{0, \dots b-1\}$
$\mathsf{seq}_b(x)$ is the representation of x over Σ_b
$\mathsf{real}_b(\sigma)$ is the rational with representation σ in base b

- (Bourke et al 2005) $x \in [0,1]$ is normal to base b iff $\mathsf{seq}_b(x)$ has finite-state dimension 1
- Therefore $\dim_{\mathrm{FS}}$ is very dependent on the alphabet
- $x \in [0,1]$ is absolutely normal iff it is normal to every base b
- Constructions of absolutely normal numbers by Turing, Schmidt, Becher, Lutz M

Effective dimension: Point to set principle

Definition (Kolmogorov complexity of x at precision δ)

$$\mathrm{K}_\delta(x) = \inf\{\mathrm{K}(\sigma) \mid |x - \mathrm{real}(\sigma)| < \delta\}$$

Definition

The *algorithmic dimension* of a point $x \in X$ is

$$\dim(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)}$$

Theorem (Lutz Lutz 2018)

Let $E \subseteq [0,1]$, *then*

$$\dim_{\mathrm{H}}(E) = \min_{A \subseteq 2^{<\omega}} \dim^A(E)$$

Can we do geometry with finite-state dimension?

Definition (M 2022)

Let T be a Σ_b-FST, $\delta > 0$ and $x \in [0,1)$. The *base-b T-information content of x at precision δ* is

$$\mathrm{K}^T_\delta(x) = \min\left\{\mathrm{K}^T(\sigma) \mid |\mathrm{real}_b(\sigma) - x| < \delta\right\}.$$

Definition (M 2022)

Let $b \geq 1$. Let $x \in [0,1)$ and $A \subseteq [0,1)$. The *base-b finite-state dimension of x* is

$$\dim^b_{\mathrm{FS}}(x) = \inf_{T\Sigma_b-\mathrm{FST}} \liminf_{\delta>0} \frac{\mathrm{K}^T_\delta(x)}{\log_b(1/\delta)},$$

the *base-b finite-state dimension of A* is

$$\dim^b_{\mathrm{FS}}(A) = \inf_{T\Sigma_b-\mathrm{FST}} \sup_{x\in A} \liminf_{\delta>0} \frac{\mathrm{K}^T_\delta(x)}{\log_b(1/\delta)}.$$

Can we do geometry with finite-state dimension?

Theorem (M 2022)

$$\dim_{\mathrm{FS}}^{b}(x) = \dim_{\mathrm{FS}}(\mathsf{seq}_b(x)),$$
$$\dim_{\mathrm{FS}}^{b}(A) = \dim_{\mathrm{FS}}(\mathsf{seq}_b(A)).$$

Can we relativize finite-state dimension?

Let T be a Σ_b-FST, $\delta > 0$ and $x \in [0,1)$

$$\mathrm{K}^{T}_{\delta}(x) = \min \left\{ \mathrm{K}^{T}(\sigma) \mid |\mathrm{real}_b(\sigma) - x| < \delta \right\}.$$

What part do we relativize here?
The enumerator real_b

$$\mathrm{K}^{T,f}_{\delta}(x) = \min \left\{ \mathrm{K}^{T}(\sigma) \mid |f(\sigma) - x| < \delta \right\}.$$

Relativized finite-state dimension

Let $f : \Sigma^{<\omega} \to [0,1]$ be such that $\mathrm{Im}(f)$ is dense

Definition

Let T be a Σ-FST, $\delta > 0$ and $x \in [0,1)$

$$\mathrm{K}_\delta^{T,f}(x) = \min\left\{\mathrm{K}^T(\sigma) \mid |f(\sigma) - x| < \delta\right\}.$$

Definition (M 2022)

Let $x \in [0,1)$ and $A \subseteq [0,1)$. The *f-relativized finite-state dimension* is

$$\dim_{\mathrm{FS}}^f(x) = \inf_{T\Sigma-\mathrm{FST}} \liminf_{\delta > 0} \frac{\mathrm{K}_\delta^{T,f}(x)}{\log_{|\Sigma|}(1/\delta)},$$

$$\dim_{\mathrm{FS}}^f(A) = \inf_{T\Sigma-\mathrm{FST}} \sup_{x \in A} \liminf_{\delta > 0} \frac{\mathrm{K}_\delta^{T,f}(x)}{\log_{|\Sigma|}(1/\delta)}.$$

Finite-state point to set principle

Theorem (M 2022)

Let $E \subseteq [0,1]$, then

$$\dim_{\mathrm{H}}(E) = \min_{f:\Sigma_2^{<\omega}\to[0,1],\,\mathrm{Im}(f)\,\mathrm{dense}} \dim_{\mathrm{FS}}^{f}(E)$$

What we can learn from this

- The oracle for which $\dim_{\mathrm{H}}(E) = \min_{A \subseteq 2^{<\omega}} \dim^{A}(E)$ requires a single (functional) query
- It can be interesting to separate compression and relativization
- The concept of optimal oracles from (Stull 2022) should be revisited for optimal enumerators

Hausdorff optimal oracles (Stull 2022)

A is an Hausdorff optimal oracle for E if

1. $\dim_{\mathrm{H}}(E) = \dim^{A}(E)$
2. For every B and $\epsilon > 0$ there is an $x \in E$ s.t.
   - $\dim^{A,B}(x) \geq \dim_{\mathrm{H}}(E) - \epsilon$
   - and for almost every $r \in \mathbb{N}$, $\mathrm{K}^{A,B}_{2^{-r}}(x) \geq \mathrm{K}^{A}_{2^{-r}}(x) - \epsilon r$

The following have Hausdorff optimal oracles,

1. $E \subseteq \mathbb{R}^n$ an analytic set
2. E with $\dim_{\mathrm{H}}(E) = \dim_{\mathrm{P}}(E)$

What to do next

- Lower bounds on $\dim_{\mathrm{FS}}^{f}(E)$
- Gambling characterization of $\dim_{\mathrm{FS}}^{f}$
- Block?-entropy characterization of $\dim_{\mathrm{FS}}^{f}$
- Gauged dimension through finite state dimension

$$\mathrm{K}_{\delta}^{T,f}(x) = \min\left\{\mathrm{K}^{T}(\sigma) \mid |f(\sigma) - x| < \delta\right\}$$

$$\dim_{\mathrm{FS},\varphi}^{f}(x) = \inf_{T\Sigma-\mathrm{FST}} \inf\left\{s \,\middle|\, \liminf_{\delta > 0} |\Sigma|^{\mathrm{K}_{\delta}^{T,f}(x)} \varphi_{s}(\delta) = 0\right\}$$

Some references

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