

# A non-trivial 3-REA Set Not Computing a Weak 3-generic

Peter M. Gerdes

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## 1 Notation & Definitions

## 2 Background

- Weak 1-genericity
- R.E. Sets and 1-genericity
- 2-genericity
- 3-genericity

## 3 3-REA Sets

- Differences From  $\Delta_3^0$  Escaping Functions
- Main Result
- Naive Strategies
- Complications

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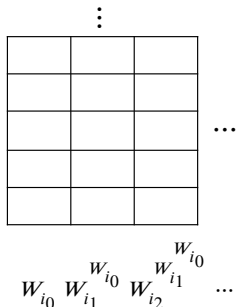
- $\sigma, \tau, \nu, \delta$  range over  $\{0, 1, \uparrow\}^{<\omega}$  (partial binary valued functions with finite domain).
- We write  $\sigma < \tau$  if  $\tau$  extends  $\sigma$  and  $\sigma < X$  if  $\sigma$  is extended by the characteristic function of  $X$ .
- $\theta$  meets  $\Gamma \subset \{0, 1, \uparrow\}^{<\omega}$  ( $\theta \Vdash \Gamma$ ) if  $(\exists \sigma \in \Gamma)(\theta \succ \sigma)$  and  $\theta$  strongly avoids  $\Gamma$  ( $\theta \Vdash \neg \Gamma$ ) if some  $(\exists \tau < \theta)(\forall \sigma \in \Gamma)(\tau \not\prec \sigma)$ .
- $f \in \omega^\omega$  dominates  $g \in \omega^\omega$  ( $f \gg g$ ) if  $(\forall^* x \in \omega)(f(x) \geq g(x))$ .
- $f$  is  $\Delta_{n+1}^0$  escaping if  $f$  isn't dominated by any  $g \leq_{\mathbf{T}} \mathbf{0}^{(n)}$

- The  $i$ -th hop is  $\mathcal{H}_i(A) \stackrel{\text{def}}{=} A \oplus W_i^A$ .
- REA sets are the result of iterating the Hop operation on  $\emptyset$ .
- The 1-REA sets are just the r.e. sets.
- The 2-REA sets are sets of the form  $W_i \oplus W_j^{W_i}$

See Jockusch and Shore [2] for a more explicit definition.

# Components as Columns

- For this talk we only care about  $n$ -REA sets up to Turing degree.
- Useful to identify the components of  $n$ -REA sets with their columns.



- In this talk we only consider the (standard) forcing relation on  $2^{<\omega}$
- $G$  is  $n$ -generic ( $n > 0$ ) if  $G \Vdash \phi$  or  $G \Vdash \neg\phi$  for all  $\Sigma_n^{0,G}$  sentences.
- Equivalently,  $G$  is  $n$ -generic if  $G$  meets or strongly avoids every  $\Sigma_n^0$  subset of  $2^{<\omega}$  (equivalently  $\{0, 1, \uparrow\}^{<\omega}$ )
- $\Gamma \subset 2^{<\omega}$  is dense if  $(\forall \tau \in 2^{<\omega})(\exists \sigma \in \Gamma)\tau < \sigma$
- $G$  is weakly  $n$ -generic if  $G$  meets every dense  $\Sigma_n^0$  subset of  $2^{<\omega}$

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## Theorem

*If  $f \in \omega^\omega$  is  $\Delta_1^0$  escaping then  $f$  computes a weak 1-generic*

- WLOG  $f$  is monotonically increasing and let  $U_i$  be  $i$ -th r.e. subset of  $2^{<\omega}$ .
- Build  $G = \lim_{n \rightarrow \infty} \tau_n$ ,  $\tau_0 = \langle \rangle$ ,  $\tau_{n+1} \succ \tau_n$ .
- Let  $\tau_{n+1} \succ \tau_n$  be in  $U_{i, f(n+1)}$  for least  $i \leq n$  or  $\tau_n$  if no such  $i$  exists.

# Verifying Weak 1-Generic

- Suppose  $U_i$  is dense but  $G$  doesn't meet  $U_i$ .
- Let  $n > 0$  large enough that  $\tau_n$  meets every  $U_j, j < i$   $G$  will ever meet.
- Suppose we can compute a bound  $l_m > |\tau_m|$  for  $m > n$ .
- Let  $h(m)$  be the least stage  $s$  such that  $U_{i,h(m)}$  includes an extension of every string of length  $l_m$ .
- If  $f(m) \geq h(m), m > n$  then  $\tau_m$  meets  $U_i$ .
- We compute  $l_m$  by assuming  $f(x) < h(x)$  for  $n < x < m$ .

Can't extend to 1-generics because we can't guarantee number of stages needed to find an extension in a non-dense  $U_i$  is computably bounded.

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## Theorem

If  $A \not\leq_T \mathbf{0}$  is r.e. then  $A$  computes a 1-generic

- The modulus for  $A$  ( $m(n) \stackrel{\text{def}}{=} \mu t (A_t \upharpoonright_{n+1} = A \upharpoonright_{n+1})$ ) is  $\Delta_1^0$  escaping.
- But we can compute full 1-generic by using the computable approximation to  $A$ .
- Same construction as before but we use stagewise approximations and allow restraint.
- Now, if we extend  $\tau_{n,s}$  to  $\tau_{n+1,s}$  to meet  $U_i$  then we preserve  $\tau_{n+1,s}$  from changes trying to meet  $U_j, j > i$

# Constructing 1-generic Below R.E.

$$m_s(n) \stackrel{\text{def}}{=} \mu t (A_t \upharpoonright_{n+1} = A_s \upharpoonright_{n+1})$$

$$r_s(i) \stackrel{\text{def}}{=} \max \{n \mid n \leq s \wedge (\exists \sigma \succ \tau_{n,s}) (\tau_{n,s} \neg \Vdash U_{i,s-1} \wedge \sigma \Vdash U_{i,s-1})\}$$

$$\bar{r}_s(i) \stackrel{\text{def}}{=} \max_{j < i} r_s(j)$$

$$i_{n+1,s}^* \stackrel{\text{def}}{=} \min_{i \leq n} \neg(\tau_{n,s} \Vdash U_{i,m_s(n)}) \wedge (\exists \sigma \succ \tau_{n,s}) (\sigma \Vdash U_{i,m_s(n+1)})$$

$$\tau_{n,s} \stackrel{\text{def}}{=} \begin{cases} \langle \rangle & \text{if } s \leq n \vee s = 0 \vee n = 0 \\ \tau_{n,s-1} & \text{unless } m_s(n+1) > m_{s-1}(n) \\ \tau_{n,s-1} & \text{if } \bar{r}_s(i_{n,s}^*) \geq n \\ \sigma & \text{o.w. where } \sigma \text{ is least witness for } i_{n,s}^* \end{cases}$$

$$G \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \lim_{s \rightarrow \infty} \tau_{n,s}$$

Note that  $\tau_{n,\infty} = \tau_{n,m(n)}$  so  $G \leq_T A$ .

# Verifying R.E. Sets Compute 1-generics

- Suppose  $i$  is least s.t.  $G \dashv \Vdash U_i \wedge G \dashv \Vdash \neg U_i$ . We show that  $A$  is computable.
- Let  $n$  large enough that  $n > \bar{r}_\infty(i)$  (exists by fact  $i$  least) and for all  $j < i$   $\tau_n \Vdash U_j \vee \tau_n \Vdash \neg U_j$  and  $t$  large enough that  $\tau_{n,t} = \tau_n$ .
- If there are  $n' \geq n, s \geq \max(t, n'), \sigma \succ \tau_{n',s}, \sigma \Vdash U_{i,s}$  then  $m(n') < s$ .
- Otherwise we'd preserve  $\tau_{n',s}$  and have  $\tau_{n',m(n')} \Vdash U_i$ .
- But, by assumption, there must be infinitetly many such  $m, s$  showing  $m \leq_T \mathbf{0}$
- Contradiction.

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## Theorem (Andrews, Gerdes and Miller)

*If  $f \in \omega^\omega$  is  $\Delta_2^0$  escaping then  $f$  computes a weak 2-generic*

- Proved in [1]. Won't prove it here.
- Idea is to try and extend to meet  $\Sigma_2^0$  sets  $\mathcal{U}_i$  by favoring those  $\sigma$  for which  $(\exists x)(\forall y)\phi(\sigma, x, y)$  appears true with least  $\max(|\sigma|, x)$ .

## Hypothesis

*If  $A \not\leq_T \mathbf{0}'$  is 2-REA then  $A$  computes a 2-generic*

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# Pattern Ends at $n = 3$

## Theorem (Andrews, Gerdes and Miller)

*There is a (pruned) perfect  $\omega$ -branching tree  $T \subset \omega^{<\omega}$ ,  $T \leq_T \mathbf{0}''$  such that if  $f \in [T]$  then  $f$  doesn't compute a weak 3-generic.*

**vertex** Node with multiple successors ( $\sigma \hat{\langle} i \rangle, \sigma \hat{\langle} j \rangle \in T, i \neq j$ ).

**$\omega$ -branching** Every vertex has infinitely many immediate successors.

**pruned** No terminal nodes (all nodes extend to paths)

**perfect** Every node is extended by a vertex.

# Pattern Ends at $n = 3$

## Theorem (Andrews, Gerdes and Miller)

*There is a (pruned) perfect  $\omega$ -branching tree  $T \subset \omega^{<\omega}$ ,  $T \leq_T \mathbf{0}''$  such that if  $f \in [T]$  then  $f$  doesn't compute a weak 3-generic.*

- No amount of (countable) non-domination suffices to compute a weak 3-generic, e.g.,  $g_j \not\gg f, j \in \omega$ .
  - View  $T$  as function on  $\omega^\omega$  by defining  $T[h]$  to be the path taking the  $h(n)$ -th option at the  $n$ -th vertex.
  - Let  $f = T[h]$  with  $h(k)$  picked large enough that  $T[h](n_k) > g_j(n_k), j \leq k$  where  $T[h] \upharpoonright_{n_k}$  is the  $k$ -th vertex along  $T[h]$
- Note that if  $f$  is monotonic and  $\Delta_{n+3}^0, n \geq 0$  escaping then  $T[f] \leq_T f \oplus \mathbf{0}''$  is as well .
  - If  $g \gg T[f]$  then  $g^*(k) = g(n_k)$  satisfies  $g^* \gg f, g^* \leq_T g \oplus \mathbf{0}''$

# Intuition Behind Failure

## Question

*What prevents the pattern from continuing indefinitely?*

- Pattern worked because more non-domination strength gave us more computational power (guessing at membership in  $\Sigma_1^0$  sets then  $\Sigma_2^0$  sets).
- But, a computable reduction can't hope to always distinguish  $\mathbf{0}^{(n)}$  big and  $\mathbf{0}^{(n+k)}$  big.
- Given finitely many potential values of  $\Phi_e(\sigma \hat{\ } \langle n \rangle)$ ,  $\mathbf{0}''$  can figure out which value is compatible with infinitely many  $n$ .
- Allows us to limit  $\Phi_e(f)$  to a narrow range of options (while allowing  $f$  to take arbitrarily large values).
- Can build  $\mathfrak{U}_e \subset 2^{<\omega}$  a dense  $\Sigma_3^0$  set  $\Phi_e(f)$  can't meet by enumerating strings outside that narrow range.

# Utility Lemma

## Lemma

Suppose for infinitely many  $l \in \omega$ ,  $\mathbf{0}''$  can enumerate  $k > 0$ ,  $\eta_i \in 2^{<\omega}$ ,  $i < 2^k - 1$ ,  $|\eta_i| \geq l + k$ . If  $f \in [T] \wedge \Phi_e(f) \downarrow \implies \Phi_e(f) \succ \eta_i$  then  $\Phi_e(f)$  isn't weakly 3-generic for any  $f \in [T]$ .

## Proof.

For each  $\sigma$  with  $|\sigma| = l$  there are  $2^k$  strings  $\tau \succ \sigma$  of length  $l + k$ . At least one of those strings  $\tau_\sigma$  must be incompatible with  $\eta_i, i < 2^k - 1$ .

For each such  $l > 0$  and  $\sigma$  with  $|\sigma| = l$  enumerate  $\tau_\sigma$  into  $\mathfrak{U}_e$ .  $\mathfrak{U}_e$  is a dense  $\Sigma_3^0$  set that isn't met by  $\Phi_e(f)$  for any  $f \in [T]$ . □

## Conditions

- A finite set  $V_s$  of vertexes ( $\cdot$ )
- For each  $\sigma \in V_s$  an infinite r.e. set of strings  $\Sigma_s(\sigma) \subset \left\{ \sigma \hat{\ } \langle n \rangle \hat{\ } \tau \mid n \in \omega, \tau \in 2^{<\omega} \right\}$
- $\theta_s^e : 2^{<\omega} \mapsto 2^{<\omega} \cup \{\uparrow\}$ ,  $e \in \omega$  such that if  $\sigma \in V_s$ ,  $\tau \in \Sigma_s(\sigma)$  then  $\Phi_e(\tau) \succ \theta_s^e(\sigma)$  (where that means  $\Phi_e(f) \uparrow$  if  $f \succ \tau$  if  $\theta_s^e(\sigma) = \uparrow$ )

$V_s$ : Nodes we commit to making  $\omega$ -branching vertexes in  $T$ .

$\Sigma_s(\sigma)$ : Possible (i.e. not in  $V_s$ ) branches extending  $\sigma$ .

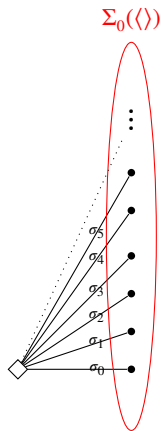
$\theta_s^e(\sigma)$ : Specifies initial segment of  $\Phi_e(\tau)$  agreed on by all  $\tau \in \Sigma_s(\sigma)$  (or that all such  $\tau$  force partiality)

## Conditions

- A finite set  $V_s$  of vertexes ( $\sigma$ )
- For each  $\sigma \in V_s$  an infinite r.e. set of strings  $\Sigma_s(\sigma) \subset \left\{ \sigma \hat{\ } \langle n \rangle \hat{\ } \tau \mid n \in \omega, \tau \in 2^{<\omega} \right\}$
- $\theta_s^e : 2^{<\omega} \mapsto 2^{<\omega} \cup \{\uparrow\}$ ,  $e \in \omega$  such that if  $\sigma \in V_s, \tau \in \Sigma_s(\sigma)$  then  $\Phi_e(\tau) \succ \theta_s^e(\sigma)$  (where that means  $\Phi_e(f) \uparrow$  if  $f \succ \tau$  if  $\theta_s^e(\sigma) = \uparrow$ )
- $V_0 = \{\langle \rangle\}$  if  $s = 0 \vee \sigma \notin V_s \vee e \geq s$  then  $\Sigma_s(\sigma) = \left\{ \sigma \hat{\ } \langle n \rangle \right\}$  and  $\theta_s^e(\sigma) = \langle \rangle$ .
- $V_{s+1} = V_s \cup \left\{ \tau_\sigma \mid \sigma \in V_s \right\}$  where  $\tau_\sigma \in \Sigma_s(\sigma)$  with  $\tau_\sigma(|\sigma|)$  large. (Hence  $|V_s| = 2^s$ ).
- $\Sigma_{s+1}(\sigma) \subset \Sigma_s(\sigma)$  and  $\theta_{s+1}^e(\sigma) \succ \theta_s^e(\sigma)$  (where  $\uparrow$  is considered  $\succ$  maximal).
- We ensure that if  $e < s, \sigma \in V_s$  then  $|\theta_s^e(\sigma)| > 2s + 1$

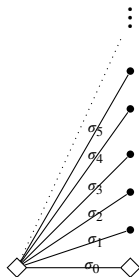


# Visualizing $T$ Construction



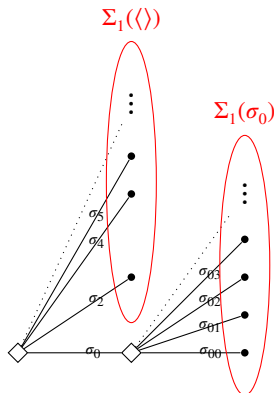
- Every  $\sigma_i \in \Sigma_0(\langle \rangle)$  has  $\Phi_e(\sigma_i) > \theta_0^e(\langle \rangle)$

# Visualizing $T$ Construction



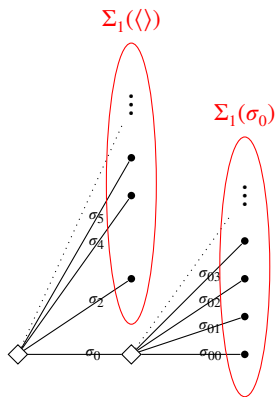
- Add new vertex in  $\Sigma_s(\tau)$  for each  $\tau \in V_s$ .

# Visualizing $T$ Construction



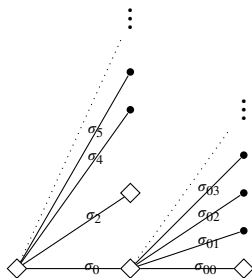
- Prune and extend (e.g. replace  $\sigma_i$  with an extension) so  
 $\sigma_i \in \Sigma_1(\langle \rangle) \implies \Phi_e(\sigma_i) > \theta_1^e(\langle \rangle)$  (now longer) and  $\Phi_e(\sigma_{0i}) > \theta_1^e(\sigma_0)$

# Visualizing $T$ Construction



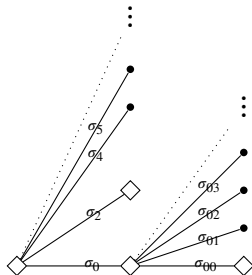
- If  $f \in [T]$  then  $\Phi_e(f) \succ \theta_1^e(\langle \rangle)$  or  $\Phi_e(f) \succ \theta_1^e(\sigma_0)$

# Visualizing $T$ Construction



- Extend each vertex with a node from allowed branches.

# Visualizing $T$ Construction



- If  $f \in [T]$  then  $\Phi_e(f) \succ \theta_2^e(\langle \rangle)$  or  $\Phi_e(f) \succ \theta_2^e(\sigma_0)$  or  $\Phi_e(f) \succ \theta_2^e(\sigma_2)$  or  $\Phi_e(f) \succ \theta_2^e(\sigma_{00})$

# Verifying Construction

- To complete proof we must only show that we can always construct  $\Sigma_{s+1}(\tau)$  from  $\Sigma_s(\tau)$  that makes  $\theta_{s+1}^e(\tau)$  sufficiently long.
- But given the length  $\mathbf{0}''$  can ask if there are infinitely many elements  $\sigma \in \Sigma_s(\tau)$  that can be extended to  $\sigma'$  with  $\Phi_e(\sigma')$  of sufficient length.
- If not remove the finitely many elements that allow convergence.
- If so  $\mathbf{0}''$  can determine which of the finitely many options for  $\Sigma_{s+1}(\tau)$  permits  $\Sigma_{s+1}(\tau)$  to be infinite.
- Repeat for each  $e < s + 1$  and  $\tau \in V_{s+1}$ .

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# Genericity From 3-REA Sets

## Question

If  $A \not\leq_T \mathbf{0}''$  is 3-REA does  $A$  compute a (weak) 3-generic?

- $A$  computes a  $\Delta_3^0$  escaping function  $m^{[3]}(x)$  (where  $m^{[n+1]}(x)$  is modulus of  $A^{[n+1]}$  over  $A^{[n]}$ ) but that's not enough.
- But several reasons to think that 3-REA sets have extra power to compute generics.
  - We get  $m^{[3]}, m^{[2]}, m^{[1]}$  with  $m^{[n]} \Delta_n^0, 1 \leq n \leq 3$  escaping. Modifications even ensure all three functions simultaneously escape a tuple  $h^1 \leq_T \mathbf{0}, h^2 \leq_T \mathbf{0}', h^3 \leq_T \mathbf{0}''$
  - Our ability to effectively approximate  $A$  offers additional power (remember non-trivial r.e. sets compute 1-generics not just weak 1-generics).
  - Approach used to build  $T$  doesn't directly translate.

# Isolating Large Values

- When we built  $T$  functionals  $\Phi_e(f)$  had to meet  $\mathcal{U}_e$  using only one large value.
  - If  $\sigma \in V_s, e < s, x \in \omega$  we could wait until we found  $\tau > \sigma \hat{\ } \langle n \rangle$  with  $\Phi_e(\tau; x)$  converging before choosing the next large value.
- Given  $A \not\leq_T \mathbf{0}''$ , 3-REA,  $k > 1$  and  $h \leq_T \mathbf{0}''$  there are infinitely many tuples  $x_0 < x_1, \dots, x_k < m^{[3]}(x_0)$  such that  $m^{[3]}(x_i) > h(x_i), i \leq k$ .
- So, infinitely often,  $\Phi_e(A; x)$  can consult  $k$  large values before trying to meet  $\mathcal{U}_e$ .

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## Theorem

*There is a 3-REA set  $A \not\leq_{\mathbf{T}} \mathbf{0}''$  that doesn't compute a weak 3-generic.*

- We know  $A$  computes a weak 2-generic
- By result in [1] every  $\Delta_3^0$  escaping function computes a 2-generic.
- Thus, result is sharp.

## Requirements

$$\mathcal{P}_i: A^{[3]}(c^i) \neq \lim_{s \rightarrow \infty} \lim_{t \rightarrow \infty} p_i(c^i, s, t)$$

$$\mathcal{Q}_{e,\sigma}: X_e \downarrow \implies [\exists \tau > \sigma](\tau \in \mathcal{U}_e \wedge \tau \notin X_e)$$

$$X_e \stackrel{\text{def}}{=} \Phi_e(A) \stackrel{\text{def}}{=} \Phi_e(A) \quad \mathcal{U}_e : \Sigma_1^0(\mathbf{0}'') \text{ subset of } 2^{<\omega}$$

$\mathcal{P}_i$  Ensures that  $A \not\leq_T \mathbf{0}''$

$\mathcal{Q}_{e,\sigma}$  Builds dense  $\mathcal{U}_e$  avoiding  $X_e$  (no other additions)

- We'll want to break these requirements up into  $\Pi_2^0$  subrequirements (to use tree method and let  $\mathbf{0}''$  see outcome).

# (Alt) Requirements

## Requirements

$$\mathcal{P}_\alpha: A^{[3]}(c^\alpha) \neq \lim_{s \rightarrow \infty} \lim_{t \rightarrow \infty} p_\alpha(c^\alpha, s, t)$$

$$\mathcal{Q}_{\alpha, \sigma}: X_\alpha \downarrow \implies [\exists \tau \succ \sigma](\tau \in \mathcal{U}_\alpha \wedge \tau \not\prec X_\alpha)$$

$$X_\alpha \stackrel{\text{def}}{=} \Phi_\alpha(A) \stackrel{\text{def}}{=} \Phi_{e_\alpha}(A) \quad \mathcal{U}_\alpha : \Sigma_1^0(\mathbf{0}'') \text{ subset of } 2^{<\omega}$$

$\mathcal{P}_\alpha$  Ensures that  $A \not\prec_T \mathbf{0}''$

$\mathcal{Q}_{\alpha, \sigma}$  Builds dense  $\mathcal{U}_\alpha$  avoiding  $X_e$  (no other additions)

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# Strategy for $\mathcal{P}_\alpha$

## Requirement

$\mathcal{P}_\alpha$ :  $A^{[3]}(c^\alpha) \neq \lim_{s \rightarrow \infty} p'_\alpha(c^\alpha, s)$       where       $p'_\alpha(c^\alpha, s) \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} p_\alpha(c^\alpha, s, t)$

## Sub-requirements

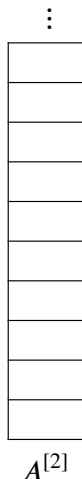
$\mathcal{P}_\alpha^k$ :       $b_k^\alpha \in A^{[2]} \iff |\{t \mid p'_\alpha(c^\alpha, t)\} = 1| > k$

- Place  $c^\alpha \in A^{[3]}$  iff  $(\exists k)(b_k^\alpha \notin A^{[2]})$
- At stage  $s$  place  $b_k$  into  $A^{[2]}$  if it's not currently in and  $|\{t \mid p_\alpha(c^\alpha, t, s)\} = 1| > k$ .
- We remove  $b_k$  at  $s_1 > s$  (by enumerating into  $A^{[1]}$ ) if  $|\{t \mid (\forall s' \in [s, s_1])(p_\alpha(c^\alpha, t, s') = 1)\}| \leq k$
- $c^\alpha \notin A^{[3]}$  if  $\lim_{s \rightarrow \infty} p'_\alpha(c^\alpha, s)$  is 1 or DNE

# First Attempt At $Q_{\alpha,\sigma}$

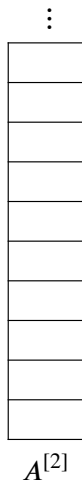
- Let's try same approach as constructing  $T$ , ensure that all 'options' for  $A$  agree on 'alot' of  $\Phi_e(A)$ .
- But  $\mathbf{0}''$  can't determine if  $c^\alpha \in A^{[3]}$ . But we can accomodate both options by agreeing on sufficently long initial segments.
- Harder problem is ensuring that  $\Phi_e(A)$  takes the same value no matter what value we get for  $\bar{k}^\alpha \stackrel{\text{def}}{=} \mu k (b_k^\alpha \notin A^{[3]})$ .
- This is analog of allowing  $f(x)$  to take on infinitely many values in construction of  $T$ .
  - (Up to  $\mathbf{0}''$  equivalence)  $\bar{k}^\alpha$  measures stage at  $c^\alpha$  enters  $A^{[3]}$
  - Effectively, we need to accomodate infinitely many options for  $m^{[3]}(c^\alpha)$ .

# Ensuring $\Phi_e(A) \succ \tau$



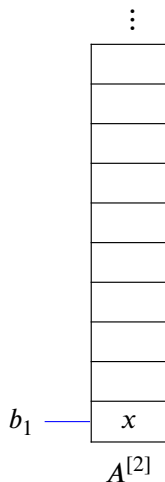
- Satisfy  $\mathcal{P}_\alpha$  allowing  $\mathbf{0}''$  to determine  $\tau$ ,  $|\tau| = 2$  with  $\Phi_e(A) \succ \tau$  assuming  $c^\alpha \in A^{[3]}$
- Try  $\tau = \langle 00 \rangle$  with highest priority, then  $\langle 01 \rangle$ ,  $\langle 10 \rangle$  and then  $\langle 11 \rangle$
- $\mathbf{0}''$  would find some other long  $\tau$  if  $c^\alpha \notin A^{[3]}$ . Easy (can only happen one way).
- Remember, elements can be removed from  $A^{[2]}$  by enumeration into  $A^{[1]}$
- Like a  $\Delta_2^0$  construction for  $A^{[2]}$  but stays out if removed infinitely many times.
- For simplicity assume totality ( $\mathbf{0}''$  will be able to check)

# Ensuring $\Phi_e(A) \succ \tau$



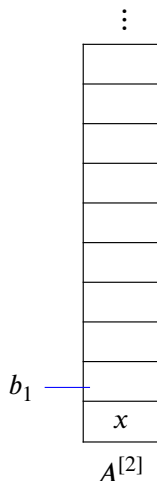
- Satisfy  $\mathcal{P}_\alpha$  allowing  $\mathbf{0}''$  to determine  $\tau, |\tau| = 2$  with  $\Phi_e(A) \succ \tau$  assuming  $c^\alpha \in A^{[3]}$
- Try  $\tau = \langle 00 \rangle$  with highest priority, then  $\langle 01 \rangle, \langle 10 \rangle$  and then  $\langle 11 \rangle$
- $\Phi_e(A_s) \succ \langle 11 \rangle$ .

# Ensuring $\Phi_e(A) \succ \tau$



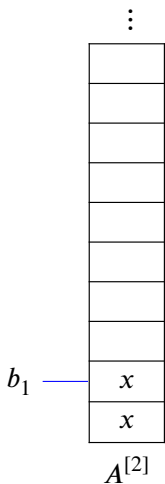
- Satisfy  $\mathcal{P}_\alpha$  allowing  $\mathbf{0}''$  to determine  $\tau$ ,  $|\tau| = 2$  with  $\Phi_e(A) \succ \tau$  assuming  $c^\alpha \in A^{[3]}$
- Try  $\tau = \langle 00 \rangle$  with highest priority, then  $\langle 01 \rangle$ ,  $\langle 10 \rangle$  and then  $\langle 11 \rangle$
- Enumerate  $b_1$ .
- $\Phi_e(A_s) \succ \langle 00 \rangle$ .

# Ensuring $\Phi_e(A) \succ \tau$



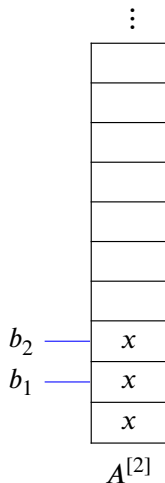
- Satisfy  $\mathcal{P}_\alpha$  allowing  $\mathbf{0}''$  to determine  $\tau$ ,  $|\tau| = 2$  with  $\Phi_e(A) \succ \tau$  assuming  $c^\alpha \in A^{[3]}$
- Try  $\tau = \langle 00 \rangle$  with highest priority, then  $\langle 01 \rangle$ ,  $\langle 10 \rangle$  and then  $\langle 11 \rangle$
- Enumerate  $b_1$ .
- $\Phi_e(A_s) \succ \langle 00 \rangle$ .
- Preserve higher priority string.
- Cancellation can only happen at  $b_k$  removing  $b_k$  and all larger enumerations.

# Ensuring $\Phi_e(A) \succ \tau$



- Satisfy  $\mathcal{P}_\alpha$  allowing  $\mathbf{0}''$  to determine  $\tau$ ,  $|\tau| = 2$  with  $\Phi_e(A) \succ \tau$  assuming  $c^\alpha \in A^{[3]}$
- Try  $\tau = \langle 00 \rangle$  with highest priority, then  $\langle 01 \rangle$ ,  $\langle 10 \rangle$  and then  $\langle 11 \rangle$
- Enumerate  $b_1$ .
- $\Phi_e(A_s) \succ \langle 10 \rangle$ .

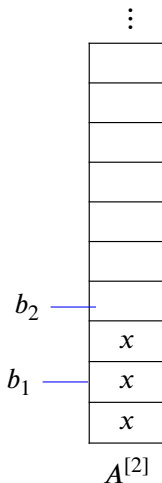
# Ensuring $\Phi_e(A) \succ \tau$



- Satisfy  $\mathcal{P}_\alpha$  allowing  $\mathbf{0}''$  to determine  $\tau$ ,  $|\tau| = 2$  with  $\Phi_e(A) \succ \tau$  assuming  $c^\alpha \in A^{[3]}$
- Try  $\tau = \langle 00 \rangle$  with highest priority, then  $\langle 01 \rangle$ ,  $\langle 10 \rangle$  and then  $\langle 11 \rangle$
- Enumerate  $b_2$ .
- $\Phi_e(A_s) \succ \langle 01 \rangle$ .

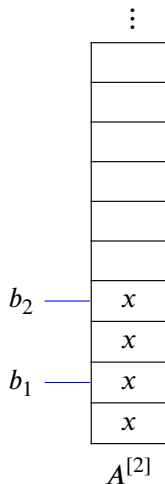


# Ensuring $\Phi_e(A) \succ \tau$



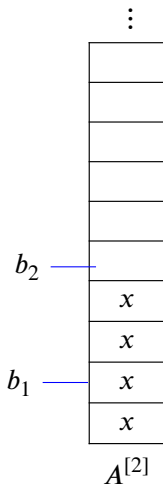
- Satisfy  $\mathcal{P}_\alpha$  allowing  $\mathbf{0}''$  to determine  $\tau$ ,  $|\tau| = 2$  with  $\Phi_e(A) \succ \tau$  assuming  $c^\alpha \in A^{[3]}$
- Try  $\tau = \langle 00 \rangle$  with highest priority, then  $\langle 01 \rangle$ ,  $\langle 10 \rangle$  and then  $\langle 11 \rangle$
- Enumerate  $b_2$ .
- $\Phi_e(A_s) \succ \langle 01 \rangle$ .
- Preserve higher priority string.
- But don't restrain/move  $b_1$  because that belongs to higher priority string  $\langle 00 \rangle$ .

# Ensuring $\Phi_e(A) \succ \tau$



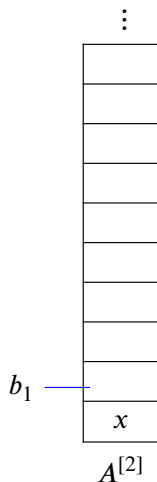
- Satisfy  $\mathcal{P}_\alpha$  allowing  $\mathbf{0}''$  to determine  $\tau$ ,  $|\tau| = 2$  with  $\Phi_e(A) \succ \tau$  assuming  $c^\alpha \in A^{[3]}$
- Try  $\tau = \langle 00 \rangle$  with highest priority, then  $\langle 01 \rangle$ ,  $\langle 10 \rangle$  and then  $\langle 11 \rangle$
- Enumerate  $b_2$ .
- $\Phi_e(A_s) \succ \langle 00 \rangle$ .

# Ensuring $\Phi_e(A) \succ \tau$



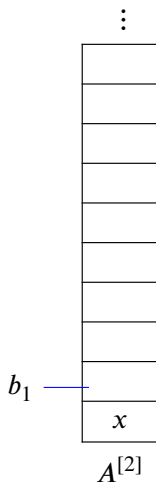
- Satisfy  $\mathcal{P}_\alpha$  allowing  $\mathbf{0}''$  to determine  $\tau$ ,  $|\tau| = 2$  with  $\Phi_e(A) \succ \tau$  assuming  $c^\alpha \in A^{[3]}$
- Try  $\tau = \langle 00 \rangle$  with highest priority, then  $\langle 01 \rangle$ ,  $\langle 10 \rangle$  and then  $\langle 11 \rangle$
- Enumerate  $b_2$ .
- $\Phi_e(A_s) \succ \langle 00 \rangle$ .
- Preserve higher priority string.
- Don't restrain/move  $b_1$  because it belongs to same string  $\langle 00 \rangle$ .

# Ensuring $\Phi_e(A) \succ \tau$



- Satisfy  $\mathcal{P}_\alpha$  allowing  $\mathbf{0}''$  to determine  $\tau, |\tau| = 2$  with  $\Phi_e(A) \succ \tau$  assuming  $c^\alpha \in A^{[3]}$
- Try  $\tau = \langle 00 \rangle$  with highest priority, then  $\langle 01 \rangle, \langle 10 \rangle$  and then  $\langle 11 \rangle$
- Later we may need to cancel  $b_1$
- But this restores state we had at earlier  $\langle 00 \rangle$  stage so  $\Phi_e(A_s) \succ \langle 00 \rangle$ .

# Ensuring $\Phi_e(A) \succ \tau$



- Satisfy  $\mathcal{P}_\alpha$  allowing  $\mathbf{0}''$  to determine  $\tau, |\tau| = 2$  with  $\Phi_e(A) \succ \tau$  assuming  $c^\alpha \in A^{[3]}$
- Try  $\tau = \langle 00 \rangle$  with highest priority, then  $\langle 01 \rangle, \langle 10 \rangle$  and then  $\langle 11 \rangle$
- If  $c^\alpha \in A^{[3]}$  then  $\Phi_e(A)$  extends highest priority  $\tau, |\tau| = 2$  seen infinitely.
- **Critically**  $\mathbf{0}''$  can determine what  $\tau$  would be if  $c^\alpha \in A^{[3]}$ .
- Doesn't affect whether (eventually) all  $b_k$  stay in  $A^{[3]}$

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# Limit May Not Exist

- Fortunately (for me), the method derived from  $T$  isn't enough.
- If the limit DNE then  $\mathbf{0}''$  never gets confirmation that  $c^\alpha \notin A^{[3]}$
- So, unlike  $T$ , we can't wait to see how  $\mathcal{P}_\alpha$  is met before starting on  $\mathcal{P}_\beta$ .
  - Requirements guessing that  $\bar{k}^\alpha = n$  (i.e. each way  $c^\alpha \in A^{[3]}$ ) can execute on cancelation of  $b_n$  (e.g. they get to know how  $\mathcal{P}_\alpha$  is met)
  - But  $\mathcal{P}_\beta$  - which guesses that  $c^\alpha \notin A^{[3]}$  - can't wait.
- If guess  $c^\alpha \notin A^{[3]}$  we do know how  $\mathcal{P}_\alpha$  is met but must work on  $\mathcal{P}_\beta$  allowing for possibility  $c^\alpha \in A^{[3]}$  with really large  $\bar{k}^\alpha$
- This is the concrete instantiation of fact that  $\Phi_e(A)$  can wait to see multiple large values before committing.

# Interference Finding $\tau < \Phi_e(A)$

- Trick to let  $\mathbf{0}''$  determine common  $\tau < \Phi_e(A)$  above can't respect both  $\mathcal{P}_\alpha$  and  $\mathcal{P}_\beta$  simultaneously.
- $\mathcal{P}_\beta$  is guessing  $c^\alpha \in A^{[3]}$  so even if  $b_m^\beta$  is cancelled infinitely often that must not cancel any  $b_k^\alpha$  infinitely many times.
- Has consequence that we can't ensure that cancelling  $b_m^\beta$  doesn't return us to a lower priority option for  $\tau$ .



# Final Trick

- Instead of ensuring that if  $b_i^\alpha$  gets cancelled we restore  $\Phi_e(A) \succ \tau$  instead ensure that if  $b_i^\alpha$  cancelled we restore  $\Phi_e(A) \succ \sigma \hat{\langle 00 \dots 0 \rangle}$  where  $|\langle 00 \dots 0 \rangle| = i$ .
- $\mathbf{0}''$  can tell if we eventually succeed at this for infinitely many  $i$ .
- If this succeeds we can (at stages we see progress) then go ahead and try to meet  $\mathcal{P}_{\beta'}$  (where  $\beta'$  guesses this succeeds) certain that when  $\mathbf{0}''$  finds out that  $b_i^\alpha \in A^{[2]}$  we can conclude  $\Phi_e(A) \succ \sigma \hat{\langle 00 \dots 0 \rangle}$ .
  - This means that even if  $\mathbf{0}''$  never sees exactly how  $\mathcal{P}_\alpha$  is satisfied we can enumerate a dense set of strings that  $\Phi_e(A)$  avoids if  $c^\alpha \in A^{[3]}$ .
- OTOH, if this fails we  $\mathbf{0}''$  discovers a string  $\sigma \hat{\langle 00 \dots 0 \rangle}$  that  $\Phi_e(A)$  avoids.
- We can try this again and again for different  $\sigma$  and interleave (in priority) with  $\mathcal{P}_\beta^k$  meaning each  $\mathcal{P}_\beta^k$  is only injured finitely many times.

- [1] Uri Andrews, Peter Gerdes, and Joseph S. Miller. “The degrees of bi-hyperhyperimmune sets”. en. In: *Annals of Pure and Applied Logic* 165.3 (Mar. 2014), pp. 803–811. ISSN: 0168-0072. DOI: 10.1016/j.apal.2013.10.004. URL: <https://www.sciencedirect.com/science/article/pii/S0168007213001528> (visited on 10/15/2021).
- [2] Carl G Jockusch and Richard A Shore. “PSEUDO JUMP OPERATORS. I: THE R. E. CASE”. en. In: *Transactions of the American Mathematical Society* 275.2 (Feb. 1983), p. 11. DOI: 10/fdstv2.