True Stages and Descriptive Set Theory

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Baire Space

We work in Baire space ω^ω (or Cantor space 2^ω).

This is a (Polish) topological space with basic clopen sets

$$[\sigma] = \{\tau \in \omega^{<\omega} : \tau \succeq \sigma\}.$$

Closed sets correspond to paths through trees.

The Borel Hierarchy

The Borel sets are the least collection of sets closed under countable intersections, countable unions, and complements.

The Borel sets can be classified by the number of intersections and unions required to construct them:

- $\mathbf{\Sigma}^0_1$: Open sets.
- $\mathbf{\Pi}^0_1$: Closed sets.
- $\mathbf{\Sigma}^0_\alpha$: Countable unions of $\mathbf{\Pi}^0_\beta$ sets for $\beta < \alpha$.
- $\mathbf{\Pi}^0_\alpha$: Countable intersections of $\mathbf{\Sigma}^0_\beta$ sets for $\beta < \alpha$.

The Difference Hierarchy

We will need two more types of sets as well:

- A set is $\mathbf{\Delta}^0_\alpha$ if it is both $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$.
- A set is $D_\eta(\mathbf{\Sigma}^0_\alpha)$ if it is a difference of η-many $\mathbf{\Sigma}^0_\alpha$ sets. E.g., if η even, of the form

$$\bigcup_{\gamma < \eta \text{ odd}} \left(U_\gamma - \bigcup_{\gamma' < \gamma} U_{\gamma'} \right)$$

  where each U_γ is $\mathbf{\Sigma}^0_\alpha$.

For example, a $D_2(\mathbf{\Sigma}^0_\alpha)$ set is of the form

$$U_1 - U_0$$

and a $D_3(\mathbf{\Sigma}^0_\alpha)$ set is of the form

$$U_2 - (U_1 - U_0).$$

The Hausdorff-Kuratowski Theorem

Theorem (Hausdorff-Kuratowski)

$$\mathbf{\Delta}^0_2 = \bigcup_\eta D_\eta(\mathbf{\Sigma}^0_1).$$

Proof.

See blackboard. □

The Hausdorff-Kuratowski Theorem

Theorem (Hausdorff-Kuratowski)

$$\mathbf{\Delta}^0_{\alpha+1} = \bigcup_\eta D_\eta(\mathbf{\Sigma}^0_\alpha).$$

If you look in Kechris, the proof is essentially:

Proof.

Let A be $\mathbf{\Delta}^0_{\alpha+1}$.

Change the topology so that A is $\mathbf{\Delta}^0_2$.

By the $\alpha = 1$ case, A is $D_\eta(\mathbf{\Sigma}^0_1)$ in the new topology.

Each $\mathbf{\Sigma}^0_1$ sets in the new topology is $\mathbf{\Sigma}^0_\alpha$ in the old topology. □

Change-of-Topology

Change-of-topology is a useful tool in descriptive set theory.

Theorem

Let $(X, \mathcal{T})$ be a Polish space with topology $\mathcal{T}$.

Let $B_1, B_2, \ldots$ be any countable collection of Borel sets in $(X, \mathcal{T})$.

There is a finer Polish topology $\mathcal{T}' \supseteq \mathcal{T}$ such that $B_1, B_2, \ldots$ are open.

Often you can also say something about the open sets in the new topology. Before, we needed that the open sets in the new topology are $\boldsymbol{\Sigma}^0_\alpha$ in the old topology.

What is this talk about?

This talk will be about a way of understanding change-of-topology in descriptive set theory using iterated true stages from computability theory.

The Effective Borel Hierarchy

The effective Borel hierarchy allows only computable unions and intersections. For α a computable ordinal:

- Σ^0_1: Effectively open sets, i.e., sets of the form $\bigcup_{\sigma \in W}[\sigma]$ for W c.e.
- Π^0_1: Effectively closed sets, i.e., paths through a computable tree.
- Σ^0_α: Unions of c.e. collections of (names for) Π^0_β sets for $\beta < \alpha$.
- Π^0_α: Intersections of c.e. collections of (names for) Σ^0_β sets for $\beta < \alpha$.

We can also define Δ^0_α, $D_\eta(\Sigma^0_\alpha)$, etc.

These hierarchies also relativize to an oracle.

Effective Descriptive Set Theory

Any $\mathbf{\Sigma}^0_\alpha$ set is $\Sigma^0_\alpha(X)$ (relative to X) for some set X. Thus it can be useful to apply effective methods even if we are not initially interested in computability.

Theorem (Hausdorff-Kuratowski, Selivanov)

$$\Delta^0_2 = \bigcup_{\eta<\omega_1^{CK}} D_\eta(\Sigma^0_1).$$

The Turing Jump

The key connection is that there is a way of thinking about $\Sigma^0_{\alpha+1}$ sets using the αth jump.

Fact

A set $A \subseteq \omega^\omega$ is $\Sigma^0_{\alpha+1}$ if and only if there is a Σ^0_1 set $V \subseteq \omega^\omega$ such that $A = \{x : x^{(\alpha)} \in V\}$.

We will use true stage constructions to approximate the jumps.

Iterated True Stage Constructions

The idea is to think of $\varnothing^{(\alpha)}$ as an iteration of the limit lemma. Each jump is a simple step, and we just need a good way to organize how they fit together.

Many computability-theoretic frameworks have been introduced to help organize this:

- Harrington worker arguments
- Lempp and Lerman's tree of strategies
- Ash and Knight's α-systems
- Montalbán's η-systems
- Greenberg and Turetsky's variation on the η-systems

Approximating the First Jump

Consider the Halting problem K.

We can computably approximate K by

$$K = \bigcup K_s$$

where K_s is the finite set containing $e < s$ if the eth program has halted at stage s.

We could think of approximating the infinite binary string K by the finite binary strings $K_s \upharpoonright s$. But it might be that every $K_s \upharpoonright s$ makes some incorrect guess.

Approximating the First Jump

The solution is Dekker non-deficiency stages.

Suppose that at each stage s, a single element n_s enters K.

Say that s is a Dekker non-deficiency stage if for all $t > s$, $n_t > n_s$. There are infinitely many non-deficiency stages.

Suppose that at stage s, we guess that $K_s \upharpoonright n_s$ is an initial segment of K. At non-deficiency stages, our guess is correct.

A stage s is 1-true if $K_s \upharpoonright n_s \prec K$.

Approximating the First Jump

A stage s is 1-true if $K_s \upharpoonright n_s \prec K$.

- There are infinitely many 1-true stages.
- If s is a 1-true stage, then it appears 1-true at every stage $t > s$.
- If s is not 1-true, then there might be stages $t > s$ which do not have enough information to see this, i.e.,

$$K_s \upharpoonright n_s \prec K_t \upharpoonright n_t.$$

We say that s appears 1-true at stage t. Such t are also not 1-true.

Approximating More Jumps

Montalbán: Iterate this through the hyperarithmetic hierarchy:

- Having approximated $\varnothing^{(\alpha)}$ at stage s by a finite string ∇^α_s, use this finite string as an oracle to approximate $\varnothing^{(\alpha+1)}$ by a finite string ∇^α_{s+1}.
- At limits, take joins.
- Use non-deficiency stages to ensure that there are infinitely many α-true stages s with $\nabla^\beta_s \prec \varnothing^{(\beta)}$ for $\beta \leq \alpha$.
- Say that s appears α-true at stage t, and write $s \leq_\alpha t$, if $\nabla^\beta_s \preceq \nabla^\beta_t$ for $\beta \leq \alpha$.
- The ∇^α_s and the relations $\leq_\alpha$ are all computable.

Disclaimer: This is all morally correct, but needs some adjustment for technical reasons.

References

For the technical details, see:

- Ash and Knight's book *Computable Structures and the Hyperarithmetical Hierarchy*
- Montalban, η-systems, in *Priority Arguments via True Stages* and *Computable Structure Theory: Beyond the arithmetic*
- Day, Greenberg, HT, Turetsky, *An effective classification of Borel Wadge classes* and *Iterated priority arguments in descriptive set theory*

Relativizing True Stages

In the true stage constructions before, we approximated $\varnothing$, $\varnothing'$, $\varnothing''$, ...

We can also relativise this to any x, approximating x, x', x'', ...

In fact, given $x \in \omega^\omega$, we can make it so that the approximation to $x^{(\alpha)}$ at stage s only depends on $x \upharpoonright s$:

- For each finite string σ and computable ordinal α, define $\sigma^{(\alpha)}$, the approximation to $x^{(\alpha)}$ for x extending σ at stage $|\sigma|$.
- Define $\sigma \leq_\alpha \tau$ if $\sigma^{(\beta)} \leq \tau^{(\beta)}$ for $\beta \leq \alpha$. We say σ appears α-true at τ.
- Say that σ is α-true for $x \in 2^\omega$, and write $\sigma \leq_\alpha x$, if $\sigma^{(\beta)} \leq x^{(\beta)}$ for $\beta \leq \alpha$.

Note that being true is now relative to the extension x.

The Structure of the Approximations

These orderings $\leq_\alpha$ on $\omega^{<\omega} \cup \omega^\omega$ have lots of nice properties:

- The relations $\leq_\alpha$, when restricted to finite strings $\omega^{<\omega}$, are computable.
- $\sigma \leq_0 \tau \Leftrightarrow \sigma \preceq \tau$.
- $\sigma \leq_\alpha \tau \Rightarrow \sigma \leq_\beta \tau$ for $\beta < \alpha$.
- for each $x \in \omega^\omega$, there infinitely many strings which are α-true for x:

$$\sigma_0 \leq_\alpha \sigma_1 \leq_\alpha \sigma_2 \leq_\alpha \cdots \leq_\alpha x.$$

- $(\omega^{<\omega}, \leq_\alpha)$ is a tree/forest.

True Stages and Topology

Some additional properties of our true stages:

- $[\sigma]_\alpha = \{\bar{x} : \sigma \leq_\alpha \bar{x}\}$ is Σ^0_α.
- Each Σ^0_α set is of the form

$$\bigcup_{\sigma \in W} [\sigma]_\alpha = \bigcup_{\sigma \in W} \{\bar{x} : \sigma \leq_\alpha \bar{x}\}$$

  for some c.e. set W.

Taking $[\sigma]_\alpha = \{\bar{x} : \sigma \leq_\alpha \bar{x}\}$ as a basis, we get a Polish topology $\mathcal{T}'$ extending the standard topology where the open sets are exactly those generated by the Σ^0_α sets.

Hausdorff-Kuratowski

This way of constructing the change of topology is particularly nice because it looks like the standard topology on ω^ω in the sense that it comes from a tree.

We can adjust our proof of the Hausdorff-Kuratowski theorem to get a proof for $\Delta^0_{\alpha+1}$ by replacing the standard tree $(\omega^{<\omega}, \leq)$ by the tree $(\omega^{<\omega}, \leq_\alpha)$.

Theorem (Hausdorff-Kuratowski, Selivanov)

For all computable α,

$$\Delta^0_{\alpha+1} = \bigcup_{\eta < \omega_1^{ck}} D_\eta(\Sigma^0_\alpha).$$

Proof.

See blackboard. □

This sounds great, but can we do anything new?

Wadge Reducibility

Definition (Wadge)

Let A and B be subsets of Baire space ω^ω.

We say that A is *Wadge reducible* to B, and write $A \leq_W B$, if there is a continuous function f on ω^ω with $A = f^{-1}[B]$, i.e.

$$x \in A \Longleftrightarrow f(x) \in B.$$

Structure of Wadge Degrees

Theorem (Martin and Monk, AD)

The Wadge order is well-founded.

Theorem (Wadge's Lemma, AD)

Given $A, B \subseteq \omega^\omega$, either $A \leq_W B$ or $B \leq_W \omega^\omega - A$.

These theorems are proved by playing a game. For Borel sets, we have Borel Determinacy without having to assume AD, and so these are always true for Borel sets.

Wadge Degrees in Second-order Arithmetic

Borel determinacy requires iterations of power-set.

Theorem (Friedman)

Borel determinacy requires ω_1 iterations of the Power Set Axiom.

Martin showed that Σ^0_4 Determinacy is not provable in second-order arithmetic.

One the other hand, one can prove that Borel Wadge games are determined in second-order arithmetic.

Theorem (Louveau and Saint-Raymond)

Borel Wadge determinacy is provable in second-order arithmetic.

Description of Wadge Degrees

There are also many comprehensive descriptions of the Borel Wadge classes:

- Louveau (1983)
- Duparc (2001)
- Selivanov, for k-partitions (2007, 2017)
- Kihara and Montalbán, for functions into a countable BQO (2019)

We use our true stage machinery to give a new description of the Borel Wadge classes, and use them to prove Borel Wadge determinacy in a reasonable fragment of second-order arithmetic.

Wadge Degrees and Reverse Math

Theorem (Day, Greenberg, HT, Turetsky)

Borel Wadge determinacy is provable in $ATR_0 + \Pi^1_1\text{–Ind}$, *and there is a complete description of the Borel Wadge classes.*

Thus the Borel Wadge degrees are semilinearly ordered and well-founded.

This simplifies Louveau and Saint-Raymond's proof in second-order arithmetic and uses a weaker subsystem. Our descriptions of the classes are inherently dynamic, and naturally lightface.

- Make a list of described classes. These are non-self-dual. Our descriptions are dynamic in nature.
- Prove a Louveau-Saint Raymond separation result for each described class Γ, which implies that if A is universal for Γ, and B is Borel, then either $A \leq_W B$ or $B \in \check{\Gamma}$, in which case $B \leq_W A^c$.
- Prove that the intersection of a described class and its dual is either a union of described classes of lower Wadge degree, like

$$\Delta^0_{\xi+1} = \bigcup_\eta D_\eta(\Sigma^0_\xi),$$

  or is a Wadge class in its own right like Δ^0_1.
- Given a Borel set, take the least described class (or dual of a described class, or $\Delta(\Gamma)$) containing it. Prove that it is complete for that class.

Theorem (Loueveau, Saint Raymond)

Suppose that Γ is a described class. Let $A \in \Gamma$. Let B_0 and B_1 be two disjoint Σ^1_1 sets. Then either:

- *There is a continuous reduction of (A, A^c) into (B_0, B_1), or*
- *There is a $\check{\Gamma}$ separator of B_0 from B_1.*

If A is universal for Γ, and B is Borel, then either $A \leq_W B$ or $B \in \check{\Gamma}$, in which case $B \leq_W A^c$.

The direct way to prove this would be to use Borel determinacy for a naturally associated game.

Louveau and Saint Raymond show by an unravelling process that there is an associated closed game.

Using true stages, we get a relatively simple description of such a game.

Theorem (Loueveau, Saint Raymond)

Suppose that Γ is a described class. Let $A \in \Gamma$. Let B_0 and B_1 be two disjoint Σ^1_1 sets. Then either:

- *There is a continuous reduction of (A, A^c) into (B_0, B_1), or*
- *There is a $\check{\Gamma}$ separator of B_0 from B_1.*

Take $\Gamma = \Sigma^0_\xi$. Let T_i be a tree whose projection is B_i.

- Player 1 plays x in A or A^c.
- Player 2 attempts to play y in B_0 (if $x \in A$) or B_1 (if $x \notin A$), with a corresponding witness f in $[T_0]$ or $[T_1]$.
- Player 2 guesses, using the true stage machinery, at whether x is in A or not. At each stage, they play an attempt at extending y and f. But they are only committed to which f they play at true stages.

Theorem (Day, Greenberg, HT, Turetsky)

Borel Wadge determinacy is provable in $ATR_0 + \Pi^1_1\text{–Ind}$*, and there is a complete description of the Borel Wadge classes.*

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References

Day, Greenberg, Harrison-Trainor, Turetsky:

- *Iterated priority arguments in descriptive set theory*
- *An effective classification of Borel Wadge classes*

Thanks!