Borel order dimension

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Outline

Notation

- $\leq$ is a **quasi order on** P if $\leq$ is a reflexive and transitive relation on P.
- $<$ is a **partial order on** P if $<$ is an irreflexive and transitive relation on P.
- A quasi order $\leq$ on P is **linear** or **total** if for any $x, y \in P$, $x \leq y \vee y \leq x$.
- A partial order $<$ on P is **linear** or **total** if for any $x, y \in P$, $x < y \vee y < x \vee x = y$.
- For a quasi order $\leq$ on P, $E_{\leq}$ is the equivalence relation on P defined by

$$p \; E_{\leq} \; q \iff (p \leq q \wedge q \leq p).$$

- For a quasi order $\leq$, $x < y$ means $x \leq y \wedge y \nleq x$. $<$ is a partial order. For a partial order $<$, $x \leqq y$ means $x < y \vee x = y$. $\leqq$ is a quasi order with $E_{\leqq} ==$.

- For a quasi order $\leq$ on P, $<$ induces a partial order on $P/E_{\leq}$, also denoted $<$.
- Example 1: $\mathcal{D} = \langle 2^{\omega}, \leq_T \rangle$, where $\leq_T$ is Turing reducibility.
- Example 2: $\langle \omega^{\omega}, \leq^{*} \rangle$, where $f \leq^{*} g$ iff $\forall^{\infty} n \in \omega \, [f(n) \leq g(n)]$.

Definition

A quasi order $\mathcal{P} = \langle P, \leq \rangle$ is called a ***Borel quasi order*** *if P is a Polish space and $\leq$ is a Borel subset of $P \times P$.*

- $\mathcal{D}$ and $\langle \omega^\omega, \leq^* \rangle$ are both Borel quasi orders.

Definition

A quasi order $\mathcal{P} = \langle P, \leq \rangle$ is said to be ***locally countable*** *(**locally finite**) if for every $x \in P$, $\{y \in P : y \leq x\}$ is countable (finite).*

- $\mathcal{D}$ is locally countable.
- $\langle \omega^\omega, \leq^* \rangle$ is not locally countable.

Definition

Suppose $\leq_0$ and $\leq$ are both quasi orders on P. $\leq$ is said to ***extend*** *$\leq_0$ if*

1. $x \leq_0 y \implies x \leq y$ *and*
2. $x\, E_{\leq_0}\, y \iff x E_{\leq} y$,

for all $x, y \in P$.
If $\leq$ is a linear quasi order which extends $\leq_0$, then we say $\leq$ ***linearizes*** *$\leq_0$.*

- The second condition is saying that $P/E_{\leq_0}$ and $P/E_{\leq}$ are the same.

Definition (Dushnik–Miller [1], 1941)

*For a quasi order $\mathcal{P} = \langle P, \leq \rangle$, the **order dimension** (or simply **dimension**) of $\mathcal{P}$ is the smallest cardinality of a collection of linear orders on $P/E_{\leq}$ whose intersection is $<$.*
$\mathtt{odim}(\mathcal{P})$ *will denote the order dimension of $\mathcal{P}$.*

Fact

The order dimension of $\mathcal{P}$ is the minimal κ such that $\langle P/E_{\leq}, < \rangle$ embeds into a product of κ many linear orders (with the coordinate wise ordering on the product).

- $\mathtt{odim}(\mathcal{P})$ is the minimal κ such that there is a sequence $\langle \leq_i : i \in \kappa \rangle$ of quasi orders on P extending $\leq$ such that for any $x, y \in P$, if $x \nleq y$, then $y <_i x$, for some $i \in \kappa$.

Elementary facts

- The dimension of a linear order is 1.
- The dimension of an antichain is 2.
- The dimension of a (set-theoretic) tree is 2.
- If $\mathcal{P}$ is an infinite quasi order, then $\mathtt{odim}(\mathcal{P}) \leq |P|$.
- If $\langle P, \leq \rangle$ embeds into $\langle Q, \leq_0 \rangle$, then $\mathtt{odim}\,(\langle Q, \leq_0 \rangle) \geq \mathtt{odim}\,(\langle P, \leq \rangle)$.

Locally finite orders

- If $\mathcal{P}$ is locally finite and $|P| = \kappa$, then $\mathcal{P}$ embeds into $\langle [\kappa]^{<\aleph_0}, \subseteq \rangle$.
- So $\mathrm{odim}(\mathcal{P}) \leq \mathrm{odim}\left(\langle [\kappa]^{<\aleph_0}, \subseteq \rangle\right)$.
- $\mathrm{odim}\left(\langle [\omega]^{<\aleph_0}, \subseteq \rangle\right)$ is $\aleph_0$.
- $\mathrm{odim}\left(\langle [\omega_1]^{<\aleph_0}, \subseteq \rangle\right)$ is $\ldots \aleph_0$.
- $\mathrm{odim}\left(\langle [\omega_2]^{<\aleph_0}, \subseteq \rangle\right)$ is $\ldots \aleph_0$.
- $\mathrm{odim}\left(\langle [\omega_3]^{<\aleph_0}, \subseteq \rangle\right)$ is $\ldots$
  1. if CH and $2^{\aleph_1} = \aleph_2$, then it is $\aleph_1$;
  2. else it is $\aleph_0$.

Theorem (Kierstead and Milner [5], 1996)

Let $\kappa \geq \omega$ be any cardinal. Then $\mathrm{odim}\left(\langle [\kappa]^{<\omega}, \subseteq \rangle\right) = \log_2(\log_2(\kappa))$.

Locally countable orders

Theorem (Higuchi, Lempp, R., and Stephan [3], 2019)

Suppose κ is any cardinal such that $\mathrm{cf}(\kappa) > \omega$ and $\mathcal{P} = \langle P, \leq \rangle$ is any locally countable quasi order of size κ^+. Then $\mathcal{P}$ has dimension at most κ.

Theorem (Kumar and Raghavan [6], 2020)

$\mathcal{D} = \langle 2^\omega, \leq_T \rangle$ has the largest order dimension among all locally countable quasi orders of size $2^{\aleph_0}$.

Theorem (Kumar and Raghavan [6], 2020)

Each of the following is consistent:

1. $\aleph_1 < \mathtt{odim}(\mathcal{D}) < 2^{\aleph_0}$;
2. $\mathtt{odim}(\mathcal{D}) = 2^{\aleph_0}$ *and* $2^{\aleph_0}$ *is weakly inaccessible;*
3. $\mathtt{odim}(\mathcal{D}) = 2^{\aleph_0} = \aleph_{\omega_1}$;
4. $\mathtt{odim}(\mathcal{D}) = 2^{\aleph_0} = \aleph_{\omega+1}$.

- Most Borel quasi orders do not have any Borel linearizations.

Definition (Harrington, Marker, and Shelah [2], 1988)

*$\mathcal{P}$ is **thin** if there is no perfect set of pairwise incomparable elements.*

Theorem (Harrington, Marker, and Shelah [2], 1988)

If $\mathcal{P} = \langle P, \leq \rangle$ is a thin Borel quasi order, then for some $\alpha < \omega_1$, there is a Borel $f : P \to 2^\alpha$ such that

1. $x \leq y \implies f(x) \leq_{\text{lex}} f(y)$ *and*
2. $x \, E_{\leq} \, y \iff f(x) = f(y)$, *for all* $x, y \in P$.

- Hence if $\langle P, \leq_0 \rangle$ is a Borel quasi order and if $\leq$ is a Borel total quasi order extending $\leq_0$, then for some $\alpha < \omega_1$, there is a Borel $f : P \to 2^\alpha$ such that

$$x \leq_0 y \implies x \leq y \implies f(x) \leq_{\text{lex}} f(y) \text{ and,}$$
$$x \, E_{\leq_0} \, y \iff x \, E_{\leq} \, y \iff f(x) = f(y),$$

for all $x, y \in P$.

Definition

On 2^ω, define the partial ordering $\leq_0$ by $x \leq_0 y$ if and only if $x = y$, or xE_0y and for the maximal $n \in \omega$ such that $x(n) \neq y(n)$, we have $x(n) < y(n)$.

- $E_{\leq_0}$ is equality and elements in distinct E_0 classes are incomparable.
- The class $[\omega \times \{0\}]_{E_0}$ is linearly ordered in type ω, $[\omega \times \{1\}]_{E_0}$ is linearly ordered in type ω^*, and every other E_0 class is linearly ordered in type $\mathbb{Z}$ by $<_0$.
- If $\langle 2^\omega, \leq_0 \rangle$ were Borel linearizable, then you could find a Borel selector for E_0, which is impossible.

Theorem (Kanovei [4], 1998)

Suppose $\langle P, \leq \rangle$ is a Borel quasi order. Then exactly one of the following two conditions is satisfied:

1. $\langle P, \leq \rangle$ *is Borel linearizable;*
2. *there is a continuous 1-1 map* $F : 2^\omega \to P$ *such that:*

   (2a) $a \leq_0 b \implies F(a) \leq F(b)$ *and*

   (2b) $a \not{E}_0\, b \implies F(a)$ *and* $F(b)$ *are* $\leq$*-incomparable.*

Borel order dimension

Definition

*Suppose $\mathcal{P} = \langle P, \leq \rangle$ is a Borel quasi order. The **Borel order dimension of** $\mathcal{P}$, denoted $\mathtt{odim}_B(\mathcal{P})$, is the minimal κ such that there is a sequence $\langle \leq_i : i \in \kappa \rangle$ of Borel quasi orders on P extending $\leq$ such that for any $x, y \in P$, if $x \not\leq y$, then $y <_i x$, for some $i \in \kappa$.*

Definition

*Let X be a set and R a binary relation on X that is disjoint from the diagonal. An **R-loop** is a finite sequence $x_0, \dots, x_k \in X$ so that $(x_i, x_{i+1}) \in R$ for all $i < k$, $(x_k, x_0) \in R$.*

Definition

*Let $\mathcal{X} = \langle X, R \rangle$ be a structure as in the previous definition. The **loop-free chromatic number of $\mathcal{X}$**, denoted $\mathcal{H}(\mathcal{X})$, is the minimal κ such that $X = \bigcup_{\lambda<\kappa} X_\lambda$, where no X_λ contains an R-loop.*

*If X is a Polish space and R is a Borel binary relation on X that is disjoint from the diagonal, then the **Borel loop-free chromatic number of $\mathcal{X}$**, denoted $\mathcal{H}_B(\mathcal{X})$, is the minimal κ such that $X = \bigcup_{\lambda<\kappa} X_\lambda$, where each X_λ is a Borel set that does not contain any R-loops.*

- Suppose $\mathcal{P} = \langle P, \leq \rangle$ is a quasi order. Let $\mathcal{A}_{\mathcal{P}} = (P \times P) \setminus \geq$ and define $\mathcal{R}_{\mathcal{P}}$ on $\mathcal{A}_{\mathcal{P}} \times \mathcal{A}_{\mathcal{P}}$ by $(p_0, q_0) \ \mathcal{R}_{\mathcal{P}} \ (p_1, q_1) \iff q_0 \leq p_1$.
- $\mathcal{R}_{\mathcal{P}}$ is disjoint from the diagonal because for any $(p, q) \in \mathcal{A}_{\mathcal{P}}$, $q \nleq p$.
- Suppose $\kappa = \mathtt{odim}(\mathcal{P})$ and that $\langle \leq_\lambda : \lambda < \kappa \rangle$ is a witness.
- Let $X_\lambda = \leq_\lambda \setminus \geq$. Then $\mathcal{A}_{\mathcal{P}} = \bigcup_{\lambda < \kappa} X_\lambda$.
- If $(p_0, q_0), \dots, (p_k, q_k)$ is an $\mathcal{R}_{\mathcal{P}}$-loop in X_λ, then $p_0 \ E_{\leq_\lambda} \ q_0$, which implies $p_0 \ E_{\leq} \ q_0$, which is impossible as $q_0 \nleq p_0$.
- Hence $\mathcal{H}\left(\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle\right) \leq \mathtt{odim}\left(\mathcal{P}\right)$.

- Conversely suppose $\mathcal{H}(\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle) = \kappa$ and that $\langle X_\lambda : \lambda < \kappa \rangle$ is a witness.
- Let $\leq_\lambda$ be the transitive closure of $\leq \cup X_\lambda$.
- $\leq_\lambda$ is then a quasi order on P and $\leq \subseteq \leq_\lambda$.
- $E_{\leq_\lambda} = E_\lambda$ because X_λ is $\mathcal{R}_{\mathcal{P}}$-loop free.
- For example, if $pX_\lambda qX_\lambda rX_\lambda p$, then $(p, q), (q, r), (r, p)$ would be an $\mathcal{R}_{\mathcal{P}}$-loop in X_λ.
- Similarly if $p \leq qX_\lambda rX_\lambda s \leq tX_\lambda p$, then $(q, r), (r, s), (t, p)$ is an $\mathcal{R}_{\mathcal{P}}$-loop in X_λ.
- If $q \nleq p$, then $(p, q) \in \mathcal{A}_{\mathcal{P}} = \bigcup_{\lambda<\kappa} X_\lambda$. So $p \leq_\lambda q$, and since $E_{\leq_\lambda} = E_{\leq}$, $p <_\lambda q$.
- Hence $\mathtt{odim}(\mathcal{P}) \leq \mathcal{H}(\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle)$
- Conclusion: $\mathtt{odim}(\mathcal{P}) = \mathcal{H}(\langle \mathcal{A}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \rangle)$.

Theorem (R. and Xiao [7])

If $\mathcal{P}$ is a Borel quasi order, then $\mathtt{odim}_B(\mathcal{P}) = \mathcal{H}_B(\langle \mathcal{A}_\mathcal{P}, \mathcal{R}_\mathcal{P} \rangle)$.

- Suppose $s = \langle n_k : k \in \omega \rangle \in \omega^\omega$ is such that $n_k \geq 2$ and $n_k \leq n_{k+1}$, for all $k \in \omega$.
- Define $T(s) = \prod_{k \in \omega} n_k$. Let D be a dense subset of $T(s)$ that intersects each level exactly once.
- For $(b_0, b_1) \in [T(s)]$, define $(b_0, b_1) \in R_0(D)$ iff there is a $d \in D$ and an $x \in \omega^\omega$, so that either:

$$b_0 = d^\frown \langle i \rangle^\frown x \text{ and } b_1 = d^\frown \langle i+1 \rangle^\frown x, or$$
$$b_0 = d^\frown \langle n_{|d|} - 1 \rangle^\frown x \text{ and } b_1 = d^\frown \langle 0 \rangle^\frown x.$$

- Let $\mathcal{G}_0(s, D) = \langle [T(s)], R_0(D) \rangle$.

Definition

$\mathcal{M} = \{M \subseteq 2^\omega : M$ *is Borel and meager*$\}$.
$\mathrm{cov}(\mathcal{M}) = \min \{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{M} \wedge 2^\omega = \bigcup \mathcal{F}\}$.

Lemma (R. and Xiao [7])

$\mathcal{H}_B (\mathcal{G}_0(s, D)) \geq \mathrm{cov}(\mathcal{M})$.

Proof.

Every Borel non-meager set must contain a loop. □

Theorem (R. and Xiao [7])

Suppose X is Polish $R \subseteq X \times X$ is Borel and disjoint from the diagonal. Then either:

1. $\mathcal{H}_B(\langle X, R\rangle) \leq \aleph_0$, *or*
2. *there exist s, D, and a continuous homomorphism* $f : \mathcal{G}_0(s, D) \to \langle X, R\rangle$.

Definition

For s and D, define $\mathcal{P}_0(s, D) = \langle [T(s)] \times 2, \leq_0 \rangle$, where $(b_0, i) \leq_0 (b_1, j)$ iff $i = 0$, $j = 1$, and $(b_0, b_1) \in R_0(D)$.

- Note that $\{((b, 1), (b, 0)) : b \in [T(s)]\} \subseteq \mathcal{A}_{\mathcal{P}_0(s,D)}$.
- Further, $((b, 1), (b, 0)) \; \mathcal{R}_{\mathcal{P}_0(s,D)} \; ((b', 1), (b', 0))$ iff $(b, 0) \leq_0 (b', 1)$ iff $b \; R_0(D) \; b'$.
- Therefore, there is a copy of $\mathcal{G}_0(s, D)$ inside the structure $\langle \mathcal{A}_{\mathcal{P}_0(s,D)}, \mathcal{R}_{\mathcal{P}_0(s,D)} \rangle$.
- Hence $\mathtt{odim}_B(\mathcal{P}_0(s, D)) \geq \mathrm{cov}(\mathcal{M})$.

Theorem (R. and Xiao [7])

For any Borel quasi order $\mathcal{P} = \langle P, \leq \rangle$ exactly one of the following holds:

1. $\mathtt{odim}_B(\mathcal{P}) \leq \aleph_0$.
2. *There exist s, D, and a continuous $f : [T(s)] \times 2 \to P$ such that:*
   - (2a) $(b_0, 0) \leq_0 (b_1, 1) \implies f((b_0, 0)) \leq f((b_1, 1))$ *and*
   - (2b) *for every $b \in [T(s)]$, $f((b, 0))$ and $f((b, 1))$ are $\leq$-incomparable.*

Corollary (R. and Xiao [7])

For every Borel quasi order $\mathcal{P}$, $\mathtt{odim}_B(\mathcal{P})$ is either countable or at least $\mathrm{cov}(\mathcal{M})$.

Theorem (R. and Xiao [7])

For every Borel quasi order $\mathcal{P}$, if $\mathtt{odim}_B(\mathcal{P})$ is countable, then $\mathcal{P}$ has a Borel linearization.

The Turing degrees

- Combining these results with my earlier results with Higuchi, Lempp, and Stephan, we get that $\mathtt{odim}_B(\mathcal{D})$ is usually strictly bigger than $\mathtt{odim}(\mathcal{D})$.
- For example, if $\mathrm{cf}(\kappa) > \omega$, $2^{\aleph_0} = \kappa^+$, and $\mathrm{MA}_\kappa(\mathrm{countable})$ holds. Then $\mathtt{odim}(\mathcal{D}) \leq \kappa < \kappa^+ = \mathrm{cov}(\mathcal{M}) = \mathtt{odim}_B(\mathcal{D})$.
- In particular, if PFA holds, then $\mathtt{odim}(\mathcal{D}) = \aleph_1 < \aleph_2 = \mathtt{odim}_B(\mathcal{D}) = 2^{\aleph_0}$.

Theorem (R. and Xiao [7])

If $\mathcal{P}$ is a locally finite Borel quasi order, then $\mathtt{odim}_B(\mathcal{P}) \leq \aleph_0$.

- Our dichotomy does not provide any natural upper bound on $\mathtt{odim}_B(\mathcal{D})$ other than $2^{\aleph_0}$.
- So it is natural to wonder weather $\mathtt{odim}_B(\mathcal{D}) = 2^{\aleph_0}$.

Theorem (R. and Xiao [7])

There is a c.c.c. forcing which forces that for every locally countable Borel quasi order $\mathcal{P}$, $\mathtt{odim}_B(\mathcal{P}) = \aleph_1$.

- So starting with a ground model $\mathbf{V}$ where $2^{\aleph_0} = \aleph_{17}$, there is a cardinal preserving forcing extension in which $2^{\aleph_0} = \aleph_{17}$ and for every locally countable Borel quasi order $\mathcal{P}$, $\mathtt{odim}_B(\mathcal{P}) = \aleph_1$.
- Each $\mathcal{P}_0(s, D)$ is locally countable. So in this model, $\mathcal{H}_B\left(\mathcal{G}_0(s, D)\right) = \aleph_1 < 2^{\aleph_0}$, for every s and D.

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