The ω-Vaught's Conjecture

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The main result

- There is a "purely structural" strengthening of Vaught's conjecture named the ω-Vaught's conjecture (ω-VC).
- Linear orders satisfy the ω-Vaught's conjecture.

Summary of the talk

1. Vaught's conjecture and the Morley analysis
2. ω-VC
3. Selected points from the proof for linear orders

Vaught's Conjecture

Conjecture: [Vaught 61] Given a first order theory over a countable vocabulary, the number of countable models of the theory is either countable or continuum.
Conjecture (infinitary version): [Vaught 61] Given a formula $\varphi \in \mathcal{L}_{\omega_1,\omega}$ over a countable vocabulary, the number of countable models of φ is either countable or continuum.

- $\mathcal{L}_{\omega_1,\omega}$ is infinitary logic; it extends first order logic by allowing countable conjunctions and disjunctions.
- Under CH the conjecture trivially holds. You can replace "continuum" with "perfectly many" to get a statement independent of set theoretic considerations.

Selected Variations on Vaught's Conjecture

Conjecture: [Martin] Given a complete, consistent first order theory T over a countable vocabulary, add a predicate for every type to create T_1. If T has fewer than $2^{\aleph_0}$ many models, then any model of T is $\aleph_0-$categorical in its T_1 theory.

Conjecture: [Becker-Kechris] For any continuous action of a Polish group on a Polish space, there are either countable or continuum many orbits.

Theorem: [Becker] One of the following holds for any complete, left invariant Polish G-space X:

- X has perfectly many orbits.
- Every orbit of X is $\mathbf{\Pi}^0_\omega$.

Complexity of $\mathcal{L}_{\omega_1,\omega}$ formulas

- $\varphi \in \mathcal{L}_{\omega_1,\omega}$ is in $\Sigma_0^{in} = \Pi_0^{in}$ if it is quantifier free and has no infinitary disjunctions or conjunctions.
- For $\alpha \in \omega_1$, φ is Σ_α^{in} if $\varphi = \bigvee\!\!\!\bigvee_i \exists(\bar{x})\psi_i(\bar{x})$ for $\psi_i \in \Pi_\beta^{in}$ with $\beta < \alpha$.
- For $\alpha \in \omega_1$, φ is Π_α^{in} if $\varphi = \bigwedge\!\!\!\bigwedge_i \forall(\bar{x})\psi_i(\bar{x})$ for $\psi_i \in \Sigma_\beta^{in}$ with $\beta < \alpha$.

Complexity of $\mathcal{L}_{\omega_1,\omega}$ formulas

- For two models M, N we say $M \leq_\alpha N$ if $\Pi^{in}_\alpha - \mathrm{Th}(M) \subseteq \Pi^{in}_\alpha - \mathrm{Th}(N)$.
- Note that $M \geq_\alpha N$ if and only if $\Sigma^{in}_\alpha - \mathrm{Th}(M) \subseteq \Sigma^{in}_\alpha - \mathrm{Th}(N)$.
- We put $M \equiv_\alpha N$ if both of the above hold.

Fact: The $\equiv_\alpha$ are Borel equivalence relations.
Theorem: [Silver 80] Borel equivalence relations have either countable or continuum many equivalence classes.

Scott rank

Theorem: [Scott] For every countable structure M there is a sentence $\varphi \in \mathcal{L}_{\omega_1,\omega}$ such that $N \cong M \iff N \models \varphi$.
Corollary: On countable structures,

$$\cong \quad = \quad \bigcap_{\alpha\in\omega_1} \equiv_\alpha$$

Definition: A φ as in the theorem statement is called a *Scott sentence*.
Definition: [Montalbán] The (parametrized) *Scott rank* of M is the least $\alpha \in \omega_1$ such that M has a $\Sigma^{in}_{\alpha+2}$ Scott sentence. We write $\mathrm{SR}(M) = \alpha$.

The Morley analysis

Theorem: [Morley] Given a formula $\varphi \in \mathcal{L}_{\omega_1,\omega}$ over a countable vocabulary, the number of countable models of φ is either countable, continuum, or $\aleph_1$.

Proof (Sketch): Let $SS(\varphi) := \{\alpha \in \omega_1 | \exists M,\ M \models \varphi \wedge \mathrm{SR}(M) = \alpha\}$ and consider cases:

1. For some $\beta < \omega_1$ there are conintuum many $\equiv_\beta$ classes.
2. $SS(\varphi)$ is bounded below some $\beta < \omega_1$. In this case, $\cong$ is $\equiv_{\beta+2}$ so is Borel. If we are not in case 1, there are only $\aleph_0$ many models.
3. $SS(\varphi)$ is cofinal in ω_1 and for all $\beta < \omega_1$ there are countably many $\equiv_\beta$ classes.This means there are $\aleph_1$ many models.

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The Vaught ordinal

Given a $\varphi \in \mathcal{L}_{\omega_1,\omega}$ we define the **Vaught ordinal**, written $\text{vo}(\varphi)$ as the least β such that either

- there are continuum many models of φ up to $\equiv_\beta$ equivalence,
- or there are only countably many models of φ and they all have Scott rank less than β.

Vaught's conjecture holds if and only if $\text{vo}(\varphi)$ is well defined for all $\mathcal{L}_{\omega_1,\omega}$ sentences φ.

Vaught ordinal examples

- Linear orders: a Π_1^{in} sentence with $\mathrm{vo}(\varphi) = 3$ as there are uncountably many $\equiv_3$ classes.
- If $\psi \in \Sigma_{\alpha+2}^{in}$ is a Scott sentence then $\mathrm{vo}(\psi) = \alpha + 1$.
- Both $\mathbb{Q}$-vector spaces and algebraically closed fields: a Π_2^{in} sentence with $\mathrm{vo}(\chi) = 3$ as they always have $\mathrm{SR}(M) < 3$.
- Boolean algebras: a Π_2^{in} sentence with $\mathrm{vo}(\theta) = \omega$ as there are uncountably many $\equiv_\omega$ classes but only countably many $\equiv_n$ classes for $n \in \omega$.

The ω-Vaught's conjecture

Conjecture: Given a formula $\varphi \in \Pi^{in}_{\alpha}$ over a countable vocabulary,

$$\text{vo}(\varphi) \leq \alpha + \omega.$$

- Because of the example of Boolean algebras, this is the best possible general bound.
- It is similar to Martin's conjecture in that we are allowing and additional ω many quantifiers to classify models.
- It is different in that it is essentially infinitary and more precisely tied to the Morley analysis and computable structure theory.
- It also gives more precise information in the "continuum case" about where the continuum is witnessed.
- It is unknown if one implies the other.

Linear orders

Theorem: [Steel 78] For any $\varphi \in \mathcal{L}_{\omega_1,\omega}$ over $\{\leq\}$ that implies all models are linear orders,

$$\text{vo}(\varphi) \leq \omega_1^{\varphi}.$$

Note that ω_1^{φ}

- is really, really big,
- is dependant on more than the complexity of the formula φ,
- uses notions from higher recursion theory.

Linear orders continued

Steel: $\mathrm{vo}(\varphi) \leq \omega_1^{\varphi}$.
Where does he need such a large ordinal?

Definition: For any $\alpha \in \omega_1$ and $x, y \in L$ a countable linear order, say

$$x \sim_\alpha y \iff \mathrm{SR}((x,y)_L) < \alpha.$$

Lemma: If $\mathrm{SR}(L) \geq \omega_1^{\varphi}$, then $L/\sim_{\omega_1^{\varphi}}$ is a dense linear order.
Proof uses Σ_1^1 bounding; is not true at non-admissible ordinals (e.g your ordering is itself an ordinal).

A better bound requires a finer combinatorial analysis of $L/\sim_\alpha$ for smaller α.

The Vaught ordinal for linear orders

Over time we preformed this analysis for $\varphi \in \Pi^{in}_{\alpha}$ whose models are linear orders:

- Version 1: $\text{vo}(\varphi) \leq (\alpha + \omega)^{\omega}$
- Version 2: $\text{vo}(\varphi) \leq \alpha \cdot \omega^{2} + \omega + 5$
- Version 3: $\text{vo}(\varphi) \leq (\alpha + \omega) \cdot 5 + \omega \cdot 5$
- Version 4: $\text{vo}(\varphi) \leq \alpha + \omega \cdot 3$
- Version 5: $\text{vo}(\varphi) \leq \alpha + \omega + 25$
- Final version: $\text{vo}(\varphi) \leq \alpha + \omega$

Theorem:[G., Montalbán] For any $\varphi \in \mathcal{L}_{\omega_1,\omega}$ over $\{\leq\}$ that implies all models are linear orders, φ satisfies ω-VC.

The main lemma

Definition: A structure M is $(\beta, \beta+\omega)$-small if for all $n \in \omega$

$$|\{B | B \equiv_\beta A\} / \equiv_{\beta+n}| \leq \aleph_0.$$

Lemma: The following are equivalent for $\varphi \in \Pi^{in}_\alpha$:

1. Every ψ that implies φ satisfies $\omega - \mathrm{VC}$.
2. For every $\beta \geq \alpha$ and $(\beta, \beta+\omega)$-small A with $A \models \varphi$ and $\mathrm{SR}(A) \geq \beta+\omega$, there is a $B \equiv_\beta A$ with $\mathrm{SR}(B) \geq \beta+\omega$ and $B \not\equiv_{\beta+\omega} A$.

Proof idea for (2) implies (1): Assume there is some $(\alpha, \alpha+\omega)$-small model of φ with a large Scott rank. Build a perfect binary tree of $\equiv_\alpha$ structures that are not $\equiv_{\alpha+\omega}$ at a given height. The set of limit structures at each path witness distinct $\equiv_{\alpha+\omega}$ classes.

What this gets us

The objective: Given a $(\beta, \beta + \omega)$-small A with $\mathrm{SR}(A) \geq \beta + \omega$, explore the space of B that have $B \equiv_\beta A$. Try to find a transformation of A into a B that satisfies the two competing goals:

1. The Scott rank of B stays at at least $\beta + \omega$,
2. B disagrees with A on some $\Pi^{in}_{\beta+n}$ formula.

The replacement lemma

Lemma: There is a non-decreasing function $f : \omega \to \omega$ which, given an $(\alpha, \alpha + \omega)$-small structure L with $\mathrm{SR}(L) \geq \alpha + n$, guarantees that there is a structure P with

$$L \equiv_{\alpha+n} P \text{ and } \alpha + n \leq \mathrm{SR}(P) \leq \alpha + f(n).$$

Idea: Apply this lemma to intervals inside of a linear ordering to control the Scott ranks of end segments.

A splitting formula

Lemma: For a fixed vocabulary, given any ordinal α, there is a $\Pi^{in}_{2\alpha+3}$ sentence ρ_α such that

$$\mathcal{A} \models \rho_\alpha \iff \mathrm{SR}(\mathcal{A}) \geq \alpha.$$

We use this idea to define $\psi_{\leq,i} := \exists x SR(L_{\leq x}) = \alpha + i$ of quantifier rank less than $\alpha + \omega$ and an analogous $\psi_{\geq,i}$.

In nearly all cases considered we construct models that disagree on some Boolean combination of the $\psi_{\leq,i}$ and $\psi_{\geq,i}$.

Fine Scott rank analysis of linear orderings

To apply the replacement lemmas effectively we need to understand how the Scott rank of suborders relate to the Scott rank of the orders they comprise.

Lemma: For any linear orderings A, B

$$\mathrm{SR}(A+B) \leq \max(\mathrm{SR}(A), \mathrm{SR}(B)) + 4.$$

Lemma: For any linear ordering A with $\mathrm{SR}(A_{\leq x}) \leq \beta$ for all $x \in A$,

$$\mathrm{SR}(A) \leq \beta + 4.$$

Combinatorial fun

The rest of the proof is purely about the combinatorics of linear orderings.

One big idea: Steel used that $L/\sim_{\omega_1^{\varphi}}$ is a dense linear order. We can reduce to the case that for some n, $L/\sim_{\alpha+n}$ with a suitable application of the replacement lemma.

While ordinals are *descriptively complicated* they are actually quite *combinatorially simple*; this is quite an important reduction.

The main takeaways

- The new result involves only relatively low-level definability and structural information about orderings.
- The use of higher recursion theory or descriptive set theory is not needed to prove VC for linear orders.
- A purely structural proof of Vaught's conjecture for other structures may be possible via ω-VC.
- Vaught's conjecture is only the beginning.
- If you think this is a straw-man, please tear it down!

Thank you!

References

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