The Borel complexity of the class of models of first-order theories

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(Joint work with Uri Andrews, David Gonzalez, Dino Rossegger, and Hongyu Zhu)

(With thanks to Ali Enayat, Roman Kossak and Albert Visser)

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Clearly, if T has an axiomatization of bounded quantifier complexity, the answer is no.
Rossegger raised this question on Mathoverflow in 2020 for the theory of true arithmetic.

Theorem (Solovay 1982, see Knight 1999)

Let T be a complete theory. Suppose $R \leq_T X$ is an enumeration of a Scott set $\mathcal{S}$, with functions t_n which are $\Delta^0_n(X)$ uniformly in n, such that for each n, $\lim_s t_n(s)$ is an R-index for $T \cap \Sigma_n$, and for all s, $t_n(s)$ is an R-index for a subset of $T \cap \Sigma_n$.
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A *Scott set* is a set $\mathcal{S} \subseteq \mathcal{P}(\omega)$ such that

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A countable model $\mathcal{M}$ *represents* a countable Scott set $\mathcal{S}$ if for all complete B_n*-types* $\Gamma(\overline{u}, x)$ and all $\overline{c} \in M$:

$$\Gamma(\overline{c}, x) \text{ realized in } \mathcal{M} \iff \Gamma \in \mathcal{S} \text{ and } \mathrm{Con}(\Gamma(\overline{c}, x) \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})).$$

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Lemma 1

Let $\mathcal{A}$ an $\mathcal{L}$-model representing a countable Scott set $\mathcal{S}$ with an enumeration $R \leq_T X$ such that $\{(i, \bar{c}) \mid R_i \text{ is the } B_1\text{-type of } \bar{c}\}$ is $\Sigma^0_2(X)$. Then there is $\mathcal{M} \cong \mathcal{A}$ with $\mathcal{M} \leq_T X$.

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Lemma 2

Let T, R, X, $\mathcal{S}$, $\{t_n\}_{n \in \omega}$ be as above. Then T has a model $\mathcal{A}$ representing $\mathcal{S}$ such that $\{(i, \overline{c}) \mid R_i \text{ is the } B_1\text{-type of } \overline{c}\}$ is $\Sigma^0_2(X)$.

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Let T, $R \leq_T X$, $\mathcal{S}$ and $\{t_n\}_{n\in\omega}$ be as in the Theorem. Then T has a model $\mathcal{M}$, representing $\mathcal{S}$, with $\mathcal{M} \leq_T X$.

Lemma 1 is proved using a finite-injury argument.
Lemma 2 is proved using a full-blown worker argument (actually, a then-new metatheorem using α-systems).

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I will now present our main theorem in an initial and then a stronger final version.

Initial Main Theorem

The family of models of true arithmetic is $\mathbf{\Pi}^0_\omega$-complete.
Indeed, for each $\mathbf{\Pi}^0_\omega$-set P, there is a continuous functional, mapping any $p \in P$ to a model of true arithmetic TA, and any $p \notin P$ to a model not satisfying Peano arithmetic PA.

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Let $T = \mathrm{TA}$. Now, more precisely, our reduction $\Gamma : p \to \mathcal{M}_p$ will be computable in $X = C \oplus (R \oplus T)'$, where C is a "Borel code" for a $\mathbf{\Pi}^0_\omega$-set P.

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The interpretation (suppressing labeling) is

$$P_n = \begin{cases} \bigcap_{j_1} \bigcup_{j_2} \cdots \bigcap_{j_n} \bigcup_{\langle j_1 \dots j_n j \rangle \in C_n} U_j & \text{if } n \text{ is odd,} \\ \bigcap_{j_1} \bigcup_{j_2} \cdots \bigcup_{j_n} \bigcap_{\langle j_1 \dots j_n j \rangle \in C_n} \overline{U_j} & \text{if } n \text{ is even.} \end{cases}$$

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4. for each n and s, $R_{t_n^p(s)} \subseteq T_p \cap \Sigma_n$.

We will indeed define $e_n^p = e_n$ to be the sequence of indices of constant functions such that $\Phi_{e_n}^{(X\oplus p)^{(n-1)}} = i_n$ for the least i_n with $T_p \cap \Sigma_n = R_{i_n}$. (This uses $X \geq_T (R \oplus T)'$.)

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Remark: We do need infinitely many theories in our theorem:
E.g., suppose we fix any complete theory $T_0 \neq \mathrm{TA}$ and ask for $\mathcal{M}_p \models \mathrm{TA}$ for $p \in P$, and $\mathcal{M}_p \models T_0$ for $p \notin P$:
TA and T_0 diverge at a fixed quantifier level.
We could reverse the roles of TA and T_0 and get that the models of TA form a $\mathbf{\Sigma}^0_\omega$-complete family.
Both facts clearly show a contradiction.

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Main Theorem

Let T be any complete theory in a relational language which is not axiomatizable by a set of axioms of bounded quantifier complexity. Then the family of models of T is $\mathbf{\Pi}^0_\omega$-complete.
Indeed, for each $\mathbf{\Pi}^0_\omega$-set P, there is a continuous functional, mapping any $p \in P$ to a model of T, and any $p \notin P$ to a model not satisfying T.

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By an observation of Rabin (1961), using Tarski's undefinability of truth in arithmetic, any completion T of PA has this property and thus the family of models of T is $\mathbf{\Pi}^0_\omega$-complete.

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Definition (Pudlák 1983, Pakhomov/Visser 2022)

A (possibly incomplete) $\mathcal{L}$-theory T is called *sequential* if it "directly interprets" *Adjunctive set theory* $\mathrm{AS}(T)$, namely, the $\{\in\}$-theory with the axioms

1. $\exists x\, \forall y\, (\neg y \in x)$ ("the empty set exists"), and
2. $\forall x\, \forall y\, \exists z\, \forall w\, (w \in z \leftrightarrow (w \in x \vee w = y))$ ("$x \cup \{y\}$ exists").

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Examples of sequential theories:
PA, $\mathrm{I}\Sigma^0_1$, PRA, $\mathrm{I}\Delta_0 + \exp$, ZF, *even* PA^- (Jeřábek 2012), $\mathrm{AS} = \mathrm{AS}(\emptyset)$ (Pakhomov/Visser 2022), but *not* Robinson's Q.

Theorem (Enayat/Visser)

For any completion T of a sequential theory in a finite language $\mathcal{L}$, $T \cap \Sigma_n$ does not axiomatize T for any n.

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We now turn to theories which have a set of axioms of bounded quantifier complexity.

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Let T be a theory and $n > 0$. Then:

- $\mathrm{Mod}(T) \in \mathbf{\Pi}^0_n$ iff T is Π_n-axiomatizable.

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- If T is Σ_n-axiomatizable, then $\mathrm{Mod}(T) \in \mathbf{\Pi}^0_{n+1}$.

We have examples that if a theory T is Σ_n-axiomatizable but not Π_n-axiomatizable then it can be $\mathbf{\Sigma}^0_n$-complete, $\mathbf{\Pi}^0_{n+1}$-complete, or in between.

Thanks!