The Borel Complexity of the Class of Models of First-order Theories: the Finite Case

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Review

Assume the language $\mathcal{L}$ is (at most) countable and relational. For any first-order theory T, consider the family $\mathrm{Mod}(T) \subseteq 2^\omega$ of all countable models of T. As the conjunction of T is an infinitary Π_ω sentence, $\mathrm{Mod}(T)$ is always $\mathbf{\Pi}^0_\omega$.

One major tool we used was the following theorem:

Theorem (Solovay)

Let T be a complete theory. Suppose $R \leq_T X$ is an enumeration of a Scott set $\mathcal{S}$, with functions t_n, $\Delta^0_n(X)$ uniformly in n, such that: for each n, $\lim_s t_n(s)$ is an R-index for $T \cap \Sigma_n$; and for all s, $t_n(s)$ is an R-index for a subset of $T \cap \Sigma_n$. Then X can compute a model $\mathcal{M} \models T$ representing $\mathcal{S}$.

Questions

All theories we consider here are assumed to be consistent, unless otherwise noted.

Definition (Boundedly Axiomatizable Theory)

A theory T is *boundedly axiomatizable* if for some $n < \omega$, T has an axiomatization consisting entirely of Π_n sentences.

We saw from last time that:

Theorem

Suppose T is complete. Then T is not boundedly axiomatizable iff $\mathrm{Mod}(T)$ *is* $\mathbf{\Pi}^0_\omega$*-complete.*

Two questions arise from this theorem:

1. What is the complexity of $\mathrm{Mod}(T)$ when T is boundedly axiomatizable?
2. What are the valid oracles for the continuous reduction?

Wadge Degree

Fact (Wadge, Martin)

The relation $\leq^*$, defined by $A \leq^* B \iff A \leq_W B \vee A \leq_W \overline{B}$, is a pre-wellordering of all Borel sets.

It remains to pinpoint which Wadge degrees are self-dual (i.e. $A \equiv_W \overline{A}$).

Fact (Steel, Van Wesep)

At successor stages, a non-self-dual Wadge degree is followed by a self-dual degree, and vice versa.
At limit stage λ, we have a self-dual degree iff $\mathrm{cf}(\lambda) = \omega$.

A more relevant criterion for us:

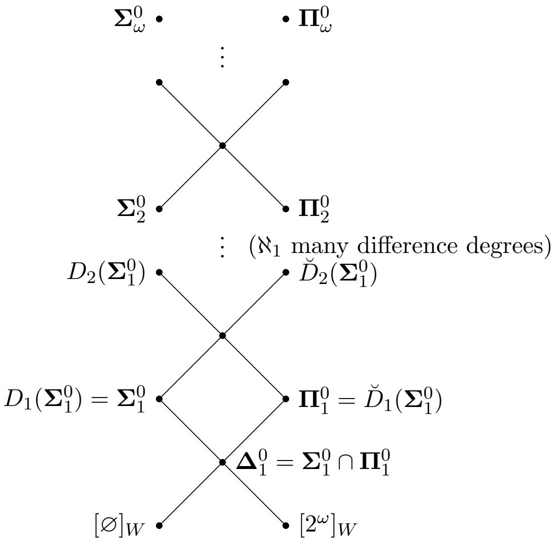
Fact

There are no $\mathbf{\Delta}^0_\alpha$-complete sets for any $\alpha \geq 2$.

Structure of the Wadge Degrees (that we care about)

Warning: Dangerous notation! For every pointclass $\mathbf{\Gamma}$ on this slide (like $\mathbf{\Sigma}^0_n, \mathbf{\Pi}^0_n$), it is more accurate to say "$\mathbf{\Gamma}$-complete."

We care about Wadge degrees below $\mathbf{\Pi}^0_\omega$. For completeness we also include $\mathbf{\Sigma}^0_\omega$.



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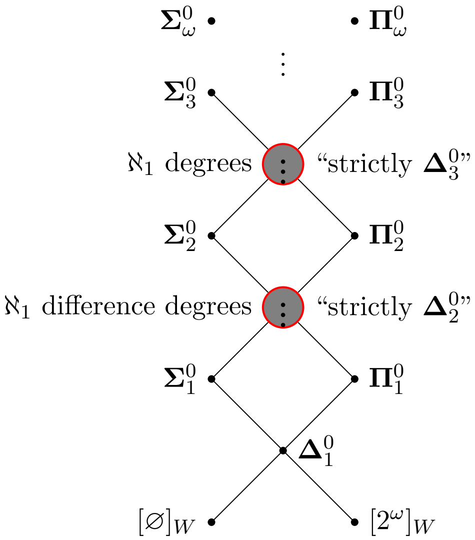

Boundedly Axiomatizable Theories

Theorem

For any $n \in \omega$ and any theory T (not necessarily complete):

1. $\mathrm{Mod}(T) \in \mathbf{\Pi}^0_n \iff T$ *is* Π_n*-axiomatizable.*
2. $\mathrm{Mod}(T) \in \mathbf{\Sigma}^0_n \Rightarrow T$ *is* Σ_n*-axiomatizable.*
3. $\mathrm{Mod}(T)$ *is* $\mathbf{\Pi}^0_n$*-complete* $\iff T$ *is* Π_n*-axiomatizable but not* Σ_n*-axiomatizable.*
4. T *is* Σ_n*-axiomatizable* $\Rightarrow \mathrm{Mod}(T) \in \mathbf{\Pi}^0_{n+1}$.

We first obtain a core lemma using Solovay's theorem.

Lemma

Suppose $n \in \omega$, and $T^+ \neq T^-$ are complete theories with $T^- \cap \Sigma_n \subseteq T^+ \cap \Sigma_n$.
Then for any $X \in \mathbf{\Sigma}^0_n$, there is a Wadge reduction f such that $f(x) \in \mathrm{Mod}(T^+)$ if $x \in X$, and $f(x) \in \mathrm{Mod}(T^-)$ otherwise.
In particular, $\mathrm{Mod}(T^+)$ is $\mathbf{\Sigma}^0_n$-hard, and $\mathrm{Mod}(T^-)$ is $\mathbf{\Pi}^0_n$-hard.

Boundedly Axiomatizable Theories

Lemma

Suppose $n \in \omega$, and $T^+ \neq T^-$ are complete theories with $T^- \cap \Sigma_n \subseteq T^+ \cap \Sigma_n$. Then for any $P \in \mathbf{\Sigma}^0_n$, there is a Wadge reduction f such that $f(x) \in \mathrm{Mod}(T^+)$ if $x \in X$, and $f(x) \in \mathrm{Mod}(T^-)$ otherwise.
In particular, $\mathrm{Mod}(T^+)$ is $\mathbf{\Sigma}^0_n$-hard, and $\mathrm{Mod}(T^-)$ is $\mathbf{\Pi}^0_n$-hard.

Proof.

Given $p \in 2^\omega$, Feed these ingredients to Solovay's theorem, noting its uniformity (like before):

- R: comes from a fixed oracle.
- t_k for $k < n$: output a fixed R-index of $T^- \cap \Sigma_k = T^+ \cap \Sigma_k$.
- t_n: check whether $p \in P$ using $p^{(n-1)}$; keep outputting index of $T^- \cap \Sigma_n$ until the Σ_n outcome (i.e. witness $p \in P$), then switch to $T^+ \cap \Sigma_n$.
- t_k for $k > n$: compute membership and then output the correct index.

Boundedly Axiomatizable Theories

Lemma

Let Γ be a family of sentences, A a set of sentences, and φ a sentence such that $A \nvdash \neg\varphi \leftrightarrow \psi$ for any $\psi \in \Gamma$. Then there are complete consistent theories $T^+ \supseteq A \cup \{\varphi\}$ and $T^- \supseteq A \cup \{\neg\varphi\}$ such that $\mathrm{Th}_\Gamma(T^-) \subseteq \mathrm{Th}_\Gamma(T^+)$.
In fact, we can take T^+ to be any completion of $A \cup \{\varphi\} \cup \mathrm{Th}_\Gamma(A \cup \{\neg\varphi\})$.

Proof.

First we verify that $A \cup \{\varphi\} \cup \mathrm{Th}_\Gamma(A \cup \{\neg\varphi\})$ is consistent: if not, then for some $\psi \in \Gamma$, we have $A \cup \{\neg\varphi\} \vdash \psi, A \cup \{\varphi\} \vdash \neg\psi$. So by deduction theorem, $A \vdash \neg\varphi \leftrightarrow \psi$, a contradiction.
Take T^+ to be any such completion. Let $\breve{\Gamma} = \{\neg\varphi | \varphi \in \Gamma\}$. By completeness, $\mathrm{Th}_\Gamma(T^-) \subseteq Th_\Gamma(T^+) \iff \mathrm{Th}_{\breve{\Gamma}}(T^+) \subseteq \mathrm{Th}_{\breve{\Gamma}}(T^-)$, so it suffices to verify $A \cup \{\neg\varphi\} \cup \mathrm{Th}_{\breve{\Gamma}}(T^+)$ is consistent. If not, then for some $\psi \in \Gamma$ we have $T^+ \vdash \neg\psi$ and $A \cup \{\neg\varphi\} \vdash \psi$. This contradicts the consistency of T^+. Thus, any completion $A \cup \{\neg\varphi\} \cup \mathrm{Th}_{\breve{\Gamma}}(T^+)$ works as T^-. □

Corollary

For any theory T and any family of sentences Γ, if T is not Γ-axiomatizable (i.e. $\mathrm{Th}_\Gamma(T)$ is not equivalent to T), then there are complete theories T_0, T_1 such that $T \subseteq T_0, T$ is inconsistent with T_1, and $\mathrm{Th}_\Gamma(T_0) \subseteq \mathrm{Th}_\Gamma(T_1)$.

Proof.

Let $A = \mathrm{Th}_\Gamma(T)$. Choose some sentence φ provable from T but not A. Check that:
(1) $A \nvdash \varphi \leftrightarrow \psi$ for any $\psi \in \Gamma$; (2) $T \cup \mathrm{Th}_{\breve{\Gamma}}(A \cup \{\neg\varphi\})$ is consistent.
For (1), if it fails then $T \vdash \varphi \leftrightarrow \psi$, so $\psi \in \mathrm{Th}_\Gamma(T) = A$. Now $A \vdash \varphi$, contradiction.
For (2), if it fails then for some $\psi \in A$, $A \cup \neg\varphi \vdash \neg\psi$, so $A \vdash \varphi$, contradiction.
Now apply the previous lemma (with $\breve{\Gamma}$ in place of Γ above) to a completion T_0 of $T \cup \mathrm{Th}_{\breve{\Gamma}}(A \cup \{\neg\varphi\})$ (as T^+) to obtain T_1 (as T^-). □

Boundedly Axiomatizable Theories

Theorem

For any $n \in \omega$ and any theory T (not necessarily complete):

1. $\mathrm{Mod}(T) \in \mathbf{\Pi}^0_n \iff T$ *is* Π_n*-axiomatizable.*
2. $\mathrm{Mod}(T) \in \mathbf{\Sigma}^0_n \Rightarrow T$ *is* Σ_n*-axiomatizable.*
3. $\mathrm{Mod}(T)$ *is* $\mathbf{\Pi}^0_n$*-complete* $\iff T$ *is* Π_n*-axiomatizable but not* Σ_n*-axiomatizable.*
4. T *is* Σ_n*-axiomatizable* $\Rightarrow \mathrm{Mod}(T) \in \mathbf{\Pi}^0_{n+1}$.

Proof.

1. $(\Rightarrow)$ Apply the corollary to T and $\Gamma = \Pi_n$ to obtain T_0, T_1. Then use the first lemma with $T^+ = T_0, T^- = T_1$.
2. Similar to the previous point: $\Gamma = \Sigma_n, T^+ = T_1, T^- = T_0$.
3. Follows directly from the previous two points.
4. Note that the conjunction of T is an infinitary Π_{n+1} sentence. □

Examples

Examples showing the Σ_n result is "tight":

Remark. Using Marker's extension, one can make these work for larger values of n.

Example

Let $\mathcal{L}$ consist of just one unary relation P, and T says P is infinite and coinfinite. Then T is Σ_1-axiomatizable and $\aleph_0$-categorical (thus complete). $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_2$-complete. [In fact, by our convention, $\mathrm{Mod}(T) \in \mathbf{\Sigma}^0_2 \Rightarrow \mathrm{Mod}(T) = \varnothing$.]

Example

$T = \mathrm{Th}(2 \cdot \mathbb{Q} + 1 + \mathbb{Q}, <, S)$ is axiomatizable by a single Σ_3 sentence and $\aleph_0$-categorical. $\mathrm{Mod}(T)$ is $\mathbf{\Sigma}^0_3$-complete.

Example

Use a 2-sorted language to combine a $\Sigma_2 - \mathbf{\Pi}^0_3$ example and a $\Sigma_3 - \mathbf{\Sigma}^0_3$ example: this gives a $\Sigma_3 - \mathbf{\Delta}^0_4$ (strict) example.

Application

While infinitary logic is more expressive than first-order logic, it does not do so more efficiently (in terms of quantifier complexity).

Theorem (Keisler 1965; Harrison-Trainor/Kretschmer 2023)

If a finitary formula φ is equivalent to an infinitary Π_n formula ψ, then φ is actually equivalent to a finitary Π_n formula.

Proof.

By compactness, it suffices to show $T = \{\varphi\}$ is Π_n-axiomatizable. This is immediate as $\mathrm{Mod}(T) = \mathrm{Mod}(\psi)$ is $\mathbf{\Pi}^0_n$. □

Effectiveness

For the continuous reduction we saw before, we seem to need the oracle $C \oplus R$, where C is the Borel code of the set we are trying to reduce, and R is a Scott set suitable for T (and each T_n).

- For the $\mathbf{\Pi}^0_\omega$ case, seem to use $C \oplus (R \oplus T \oplus (\bigoplus_n T_n))'$ directly?
- Is this necessary, especially R?

Definition (Effective Wadge Reducibility)

$X \subseteq \omega^\omega$ *effectively Wadge reduces* to $Y \subseteq \omega^\omega$ if there exists a Turing operator Φ such that for every Borel code C of X and every $x \in \omega^\omega$, $x \in X \iff \Phi^{C \oplus x} \in Y$. If C can be dropped above, we say X is *computably reducible* to Y.

Effectiveness

Theorem

There are complete theories T with $\mathrm{Mod}(T)$ $\mathbf{\Pi}^0_\omega$-complete under Wadge reducibility, but not $\mathbf{\Sigma}^0_2$-hard under effective Wadge reducibility.

In fact T can taken to be any completion of $I\exists^{\leq}_1$, induction for bounded existential formulas (in the language of arithmetic). Such theories are important because they exhibit the Tennenbaum phenomenon:

Fact (Wilmers)

Any nonstandard model of $I\exists^{\leq}_1$ is not computable.

Definition

$I(T) \subseteq \omega$ is the set of all indices of computable functions computing a model of T.

Effectiveness

Lemma

If $\mathrm{Mod}(T)$ *is* $\mathbf{\Sigma}^0_n$*-hard for computable reducibility then* $I(T)$ *is* Σ^0_n*-hard for* m*-reduction.*

Proof.

Consider a Σ^0_n set $S \subseteq \omega$ and define $C(S) = \{X \in \omega^\omega | X(0) \in S\} \subseteq \omega^\omega$, which is $\mathbf{\Sigma}^0_n$. Let Φ be an effective Wadge reduction from $C(S)$ to $\mathrm{Mod}(T)$. Let $\mathbf{x} \in \omega^\omega$ be given by $\mathbf{x} = (x, 0, 0, 0, \cdots)$. Define a computable function $f : \omega \to \omega$ where $f(x)$ is an index of the computable set $\Phi(\mathbf{x})$. Then f is an m-reduction from S to $I(T)$, as

$$x \in S \iff \mathbf{x} \in C(S) \iff \Phi(\mathbf{x}) \in \mathrm{Mod}(T) \iff f(x) \in I(T).$$

□

Effectiveness

Lemma

If T is a consistent theory that contains $I\exists_1^{\leq}$, then $I(T)$ is not Σ_2^0-hard for m-reduction.

Proof.

If not, for a given Σ_2^0-complete set A there is a computable function f such that for all $n \in \omega, \Phi_{f(n)} \models T \iff n \in A$. If T is not contained in the theory of true arithmetic, then T has no computable model, a contradiction. Otherwise, the above shows that $I(\mathbb{N})$ is Σ_2^0-hard. However $I(\mathbb{N})$ is Π_2, as $\mathbb{N}$ has a computable infinitary Π_2 Scott sentence, a contradiction. □

Combining the previous two lemmas immediately gives the theorem, noting that T cannot be boundedly axiomatizable by Enayat/Visser.

Further Questions

- Characterize the Wadge degrees (and the difference degrees) occupied by Mod(T)?
  In particular, how do they differ from the degrees that are Scott complexities?
- Can more be said about the $\mathbf{\Pi}^0_\omega$ case when T is incomplete (and not sequential)?
- More analysis on oracles?

Bibliography

[1] Uri Andrews et al. *The Borel complexity of the class of models of first-order theories.* In preparation.

[2] Ali Enayat and Albert Visser. *Incompleteness of boundedly axiomatizable theories.* 2024. arXiv: 2311.14025 [math.LO].

[3] Alexander S. Kechris. *Classical descriptive set theory.* Vol. 156. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. xviii+402. ISBN: 0-387-94374-9. DOI: 10.1007/978-1-4612-4190-4. URL: https://doi.org/10.1007/978-1-4612-4190-4.

Thank you for listening!