Pinned Distance Sets Using Effective Dimension

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Distance sets

Let $E \subseteq \mathbb{R}^n$. The distance set of E is

$$\Delta E = \{|x - y| \mid x, y \in E\}.$$

More generally, if $x \in \mathbb{R}^n$, the pinned distance of E w.r.t. x is

$$\Delta_x E = \{|x - y| \mid y \in E\}.$$

Distance sets

When E is a finite set, Erdös conjectured that $|\Delta E|$ is at least (almost) linear in terms of $|E|$.

- In a breakthrough paper, Guth and Katz proved this in the plane.
- Still an important open problem for $\mathbb{R}^n$ with $n \geq 3$.

Falconer posed an analogous question for the case that E is infinite, known as Falconer's *distance set problem*.

- If $E \subseteq \mathbb{R}^n$ has $\dim_H(E) > n/2$, then ΔE has positive measure.
- Still open in all dimensions.
- Guth, Iosevich, Ou and Wang, proved that if $E \subseteq \mathbb{R}^2$ and $\dim_H(E) > 5/4$, then $\mu(\Delta E) > 0$.

Distance sets

Substantial progress has been made in a slightly different direction, on the Hausdorff dimension of *pinned distance sets* in the plane for "many" $x \in E$.

- Shmerkin proved that, if $\dim_H(E) > 1$ and $\dim_H(E) = \dim_P(E)$, then $\dim_H(\Delta_x E) = 1$.
- Liu showed that, if $\dim_H(E) = s \in (1, 5/4)$, then $\dim_H(\Delta_x E) \geq \frac{4}{3}s - \frac{2}{3}$.
- Shmerkin improved this bound when $\dim_H(E) = s \in (1, 1.04)$, by proving that
$$\dim_H(\Delta_x E) \geq 2/3 + 1/42 \approx 0.6904$$
- S. proved that, for any E with $\dim_H(E) > 1$,
$$\dim_H(\Delta_x E) \geq \frac{\dim_H(E)}{4} + \frac{1}{2}$$

Our results

Let $E \subseteq \mathbb{R}^2$ be analytic and $1 < d < \dim_H(E)$.

- There is a subset F of full dimension such that, for all $x \in F$,
$$\dim_H(\Delta_x E) \geq \frac{d(d-4)}{d-5}$$
  - This improves the best known bounds when $\dim_H(E) \in (1, 1.127)$.
- For all x outside a set of dimension 1
$$\dim_H(\Delta_x E) \geq \frac{\dim_P(E)+1}{2\dim_P(E)}.$$
- If $\dim_P(E) < \frac{d(3+\sqrt{5})-1-\sqrt{5}}{2}$, then for all x in a subset of full dimension $\dim_H(\Delta_x E) = 1$.
- There is a point $x \in E$ such that
$$\dim_P(\Delta_x E) \geq \frac{12-\sqrt{2}}{8\sqrt{2}} \approx 0.9356$$
  - Improves (slightly) the Keleti-Shermkin bound for packing dimension.

Regularity results

We can generalize the problem, by considering a *pinned set* X, and a *test set* Y, and investigating

$$\sup_{x \in X} \dim_H(\Delta_x Y).$$

We proved that, under some regularity assumptions, the distance sets achieve maximal dimension.

1. If Y is analytic, with $\dim_H(Y) > 1$ and $\dim_P(Y) < 2\dim_H(Y) - 1$. Then, for any subset X with $\dim_H(X) > 1$,
$$\dim_H(\Delta_x Y) = 1.$$
for all $x \in X$ outside a set of (Hausdorff) dimension one.
2. If Y is analytic with $\dim_H(Y) > 1$, and X satisfies $\dim_H(X) = \dim_P(X) > 1$, then there is a subset $F \subseteq X$ such that,
$$\dim_H(\Delta_x Y) = 1,$$
for all $x \in F$.

Kolmogorov complexity

Fix a universal Turing machine U. Let $x \in \mathbb{R}$ and $r \in \mathbb{N}$. The **Kolmogorov complexity of x at precision r** is

$$K_r(x) = \text{minimum length input } \pi \in \{0,1\}^* \text{ such that } U(\pi) = x{\restriction}r,$$

where $x{\restriction}r$ is the first r bits in the binary representation of x.

- Can think of U as a computer.
- Can think of π as a program written in, e.g., Python.

Let $y \in \mathbb{R}$. The **Kolmogorov complexity of x at precision r given y at precision s** is

$$K_{r,s}(x \mid y) = \text{minimum length input } \pi \in \{0,1\}^* \text{ such that } U(\pi, y{\restriction}s) = x{\restriction}r.$$

Kolmogorov complexity

- For every $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$, $0 \leq K_r(x) \leq nr + O(\log r)$.
    - If x is rational, then $K_r(x) = O(\log r)$
    - Almost every point satisfies $K_r(x) = r - O(\log r)$ for every $r \in \mathbb{N}$. We call these points *random*.
- Symmetry of information: For every $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $r, t \in \mathbb{N}$,
$$K_{r,t}(x, y) = K_t(y) + K_{r,t}(x \mid y) + O(\log r + \log t).$$
- We can *relativize* the definitions in the natural way to get $K_r^A(x), K_{r,t}^A(x \mid y), \ldots$ for any oracle $A \subseteq \mathbb{N}$.

Effective dimension

Let $x \in \mathbb{R}^n$. The *(effective Hausdorff) dimension of* x is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}.$$

The *(effective) packing dimension of* x is

$$\operatorname{Dim}(x) = \limsup_{r \to \infty} \frac{K_r(x)}{r}.$$

- $0 \leq \dim(x) \leq \operatorname{Dim}(x) \leq n$.
- Almost every point satisfies $\dim(x) = \operatorname{Dim}(x) = n$.
- If x is rational, then $\dim(x) = \operatorname{Dim}(x) = 0$.

Point-to-set principle

Theorem (J. Lutz and N. Lutz)

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x) \text{ and } \dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \operatorname{Dim}^A(x).$$

- The Hausdorff dimension of a *set* is characterized by the (effective) dimension of the *points* in the set.
- Allows us to use computability to attack problems in geometric measure theory.

Pinned distance sets using effective dimension

Theorem (Fiedler, S.)

Let $X, Y \subseteq \mathbb{R}^2$ *such that* Y *is analytic, with* $1 < \dim_H(Y), \dim_H(X)$. *Then there is a subset* $F \subseteq X$ *of full dimension, such that, for all* $x \in F$,

$$\dim_H(\Delta_x Y) \geq d\left(1 - \frac{(D-1)(D-d)}{2(D^2+D-1)-2d(2D-1)}\right),$$

where $d = \min\{\dim_H(X), \dim_H(Y)\}$ *and* $D = \max\{\dim_P(X), \dim_P(Y)\}$.

We reduce our main theorem on the Hausdorff dimension of pinned distance sets to this pointwise analog.

1. Orponen's theorem on radial projections.
2. Point-to-set principle.

Pinned distance sets using effective dimension

Theorem (Fiedler, S.)

Suppose that $x, y \in \mathbb{R}^2$, $e_1 = \frac{y-x}{|y-x|}$ *satisfy the following.*

(C1) $\dim(x), \dim(y) > 1$

(C2) $K^x_r(e_1) \approx r$ *for all* r.

(C3) $K^x_r(y) \approx K_r(y)$ *for all sufficiently large* r.

(C4) $K_r(e_1 \mid y) \approx r$ *for all* r.

Then

$$\dim^x(|x-y|) \geq d\left(1 - \frac{(D-1)(D-d)}{2(D^2+D-1)-2d(2D-1)}\right),$$

where $d = \min\{\dim(x), \dim(y)\}$ *and* $D = \max\{\mathrm{Dim}(x), \mathrm{Dim}(y)\}$.

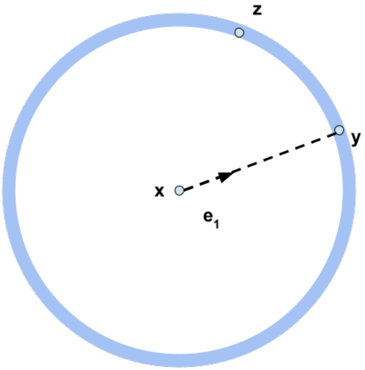
Pointwise setting

We fix $x, y \in \mathbb{R}^2$ satisfying the conditions. To get a lower bound on $\dim^x(|x - y|)$, we show the analogous bound at each precision (scale). Fix a precision $r \in \mathbb{N}$.

1. Symmetry of information: proving a lower bound on $K^x_r(|x - y|)$ is equivalent to establishing an upper bound on $K^x_r(y \mid |x - y|)$.
   - How many bits are needed to specify (a 2^{-r}-approximation of) y if you know (2^{-r}-approximations of) x and $|x - y|$.
2. A 2^{-r}-approximation of $|x - y|$ gives an annulus of thickness 2^{-r}. We need to specify where y is on this annulus (trivial bound - give r bits).
3. Use an induction on scales approach - find a "nice" sequence of precisions $r_1 < r_2 < \ldots r_k = r$. To specify y to precision r, first specify to precision r_1, using this to specify y to precision r_2, and so on.
4. We choose our precisions based on the behavior of the complexity function $K_s(y)$.
   - When the complexity is growing very quickly (slope at least 1), we are able to show that $K(|x - y|)$ is growing at a rate of 1 (best possible).

Complexity of y increases slowly

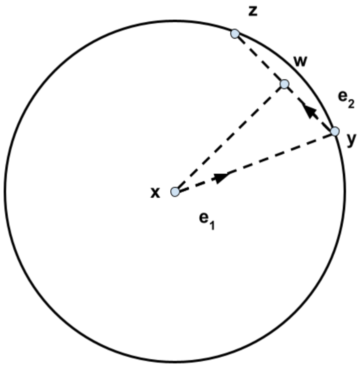
Suppose that between precisions r_i and r_{i+1}, the complexity of y is increasing slowly (much less than slope 1)



- Want to specify y up to precision r_{i+1}, given x and $|x - y|$ and given y up to precision r_i.
- Want to show that y is essentially the only point in the annulus whose complexity is growing slowly.
- That is, if z is in the annulus, then either z is very close to y, or the complexity of z is growing quickly.

Complexity of y increases slowly

Goal: Prove the complexity of other points on annulus are growing more quickly than that of y.



For any $e \in \mathcal{S}^1$ and $x \in \mathbb{R}^2$, $p_e x = e \cdot x$.

- Reduce this to *projections*.
- Main idea: we can compute x if we know y, z and the position of x along the line with direction $e_2^\perp$ containing x.

$$K(x \mid y) \lesssim K(z \mid y) + K(x \mid p_{e_2}x, e_2)$$

- Thus, if complexities of z and y are growing very slowly, then the complexity of x is growing slowly.
- Goal is to prove that $K(x \mid p_{e_2}x, e_2)$ is small.

Reducing to projections

Suppose that $|x - z| = |x - y|$, and let $s := -\log \|y - z\|$.

$$\begin{aligned} K_{r-s}(x \mid y) &\lesssim K_r(z \mid y) + K_{r-s,r-s,r,r}(w, e_2 \mid y, z) + K_{r-s,r,r}(x \mid w, e_2) \\ &\lesssim K_r(z \mid y) + K_{r-s}(x \mid p_{e_2}x, e_2). \end{aligned}$$

We know that $K_{r-s}(x \mid y) \gtrsim d(r - s)$. So, if

- the complexity of z increases at least as slowly as that of y, and
- we can get a strong enough upper bound on $K_{r-s}(x \mid p_{e_2}x, e_2)$

we have a contradiction - i.e., no such z exists on the annulus.

Projection theorem

- We want to bound $K_r(x \mid p_e x, e)$ - the complexity of computing (an approximation of) x given (approximations of) $p_e x$ and e.
- When the direction e is random relative to x, i.e., $K_r^x(e) \approx r$, we know that $K_r(x \mid p_e x, e) \approx K_r(x) - r$.
  - This is the pointwise analog of Marstrand's projection theorem.
- Unfortunately we don't have enough control over the direction to directly apply this result.
- However, we **do** have enough control to ensure that e is random *up to some initial precision*:

$$K_s^x(e) \approx s,$$

where $s = -\log|z - y|$.

Projection theorem

Theorem (Fiedler, S.)

Let $x \in \mathbb{R}^2$, $e \in \mathcal{S}^1$, $\varepsilon \in \mathbb{Q}^+$, $C \in \mathbb{N}$, $A \subseteq \mathbb{N}$, and $t, r \in \mathbb{N}$. Suppose that r is sufficiently large, and that the following hold.

(P1) $1 < d \leq \dim^A(x) \leq \operatorname{Dim}^A(x) \leq D$.

(P2) $t \geq \frac{d(2-D)}{2} r$.

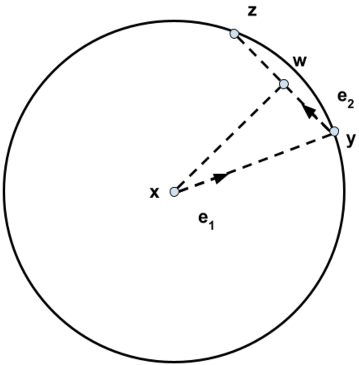
(P3) $K_s^{x,A}(e) \geq s - C \log s$, *for all* $s \leq t$.

Then

$$K_r^A(x \mid p_e x, e) \leq \max\{\frac{D-1}{D}(dr - t) + K_r^A(x) - dr, K_r^A(x) - r\} + \varepsilon r.$$

Complexity of y increases slowly

Goal: Prove the complexity of other points on annulus are growing more quickly than that of y.



For any $e \in \mathcal{S}^1$ and $x \in \mathbb{R}^2$, $p_e x = e \cdot x$.

- We can compute x if we know y, z and the position of x along the line with direction $e_2^{\perp}$ containing x.

  $$K(x \mid y) \lesssim K(z \mid y) + K(x \mid p_{e_2} x, e_2)$$

- Thus, if complexities of z and y are growing sufficiently slowly, we have a contradiction.

The End

Thank you!