Algorithmic Fractal Dimensions

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Hilbert's Final Test

The final test of every new theory is its success in answering preexistent questions that the theory was not specifically created to answer.

David Hilbert (1925)

Our topic today

How algorithmic fractal dimensions passed Hilbert's final test

Algorithmic Information (Kolmogorov Complexity)

The *Kolmogorov complexity* of a string $x \in \{0,1\}^*$ is

$$\mathrm{K}(x) = \min\left\{|\pi| \,\middle|\, \pi \in \{0,1\}^* \text{ and } U(\pi) = x\right\},$$

where U is a universal Turing machine.

- It matters little (small additive constant) which U is chosen for this.
- $\mathrm{K}(x) =$ amount of *algorithmic information* in x.
- $\mathrm{K}(x) \leq |x| + o(|x|)$.
- x is "random" if $\mathrm{K}(x) \approx |x|$.
- Routine coding extends this to $\mathrm{K}(x)$ for $x \in \mathbb{N}$, $x \in \mathbb{Q}$, $x \in \mathbb{Q}^n$, etc.

Dimensions of Points

Work in Euclidean space $\mathbb{R}^n$.

The *Kolmogorov complexity* of a set $E \subseteq \mathbb{Q}^n$ is

$$\mathrm{K}(E) = \min\{\mathrm{K}(q) \mid q \in E\}\,.$$

(Shen and Vereshchagin 2002)

The Kolmogorov complexity of a set $E \subseteq \mathbb{R}^n$ is

$$\mathrm{K}(E) = \mathrm{K}(E \cap \mathbb{Q}^n)\,.$$

Note that

$$E \subseteq F \implies \mathrm{K}(E) \geq \mathrm{K}(F)\,.$$

Dimensions of Points

Let $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$. The *Kolmogorov complexity* of x at *precision* r is

$$\mathrm{K}_r(x) = \mathrm{K}\big(B_{2^{-r}}(x)\big) \,,$$

i.e., the number of bits required to specify some rational point $q \in \mathbb{Q}^n$ such that $|q - x| \leq 2^{-r}$.

Dimensions of Points

For $x \in \mathbb{R}^n$,

$$\dim(x) = \liminf_{r \to \infty} \frac{\mathrm{K}_r(x)}{r} .$$

Easy fact. $0 \leq \dim(x) \leq n$, and there are uncountably many points of each dimension in this interval.

Old fact (J. Lutz '00 + Hitchcock '03). If $E \subseteq \mathbb{R}^n$ is a union of Π^0_1 sets, then

$$\dim_{\mathrm{H}}(E) = \sup_{x \in E} \dim(x) .$$

classical Hausdorff (fractal) dimension → $\dim_{\mathrm{H}}(E)$

dimensions of individual points → $\dim(x)$

∴ Dimensions of points are geometrically meaningful.

Dimensions of Points

For $x \in \mathbb{R}^n$,

$$\mathrm{Dim}(x) = \limsup_{r \to \infty} \frac{\mathrm{K}_r(x)}{r} . \qquad \text{(strong dimension)}$$

$\dim(x)$ is the "Σ^0_1 version" of $\dim_{\mathrm{H}}$. (Hausdorff dimension)

$\mathrm{Dim}(x)$ is the "Σ^0_1 version" of $\dim_{\mathrm{P}}$. (packing dimension)

Point-to-Set Principle

Theorem (J. Lutz and N. Lutz, 2018)

For every $E \subseteq \mathbb{R}^n$,

$$\dim_{\mathrm{H}}(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x)\,.$$

$\therefore$ In order to prove a lower bound

$$\dim_{\mathrm{H}}(E) \geq \alpha\,,$$

it suffices to show that

$$(\forall A \subseteq \mathbb{N})(\forall \varepsilon > 0)(\exists x \in E)\, \dim^A(x) \geq \alpha - \varepsilon$$

or, if you're lucky, that

$$(\forall A \subseteq \mathbb{N})(\exists x \in E)\, \dim^A(x) \geq \alpha\,.$$

Point-to-Set Principle

Theorem (J. Lutz and N. Lutz, 2018)

For every $E \subseteq \mathbb{R}^n$,

$$\dim_{\mathrm{P}}(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x)\,.$$

Fractal geometers have studied *local dimensions* (a.k.a. *pointwise dimensions*) at least since the 1930s.

Recall: An *outer measure* on a set X is a function

$$\mu \mathcal{P}(X) \to [0, \infty]$$

with the following three properties.

(i) $\mu(\emptyset) = 0$.

(ii) $E \subseteq F \implies \mu(E) \leq \mu(F)$.

(iii) $\mu\left(\bigcup_{n=0}^{\infty} E_n\right) \leq \sum_{n=0}^{\infty} \mu(E_n)$.

Definition

If μ is a finite outer measure on $\mathbb{R}^n$, then the *lower* and *upper local dimensions* of μ at a point $x \in \mathbb{R}^n$ are

$$\dim_{\mathrm{loc}} \mu(x) = \liminf_{r \to \infty} \frac{\log \frac{1}{\mu(B(x, 2^{-r}))}}{r}$$

and

$$\mathrm{Dim}_{\mathrm{loc}} \mu(x) = \limsup_{r \to \infty} \frac{\log \frac{1}{\mu(B(x, 2^{-r}))}}{r},$$

respectively.

Are these classical local dimensions related to the algorithmic fractal dimensions that we have defined?

Answer (N. Lutz 2017): Yes, with a very nonclassical choice of the outer measure!

Definition

For $E \subseteq \mathbb{R}^n$,

$$\boldsymbol{\kappa}(E) = 2^{-\mathrm{K}(E)}.$$

Theorem (N. Lutz 2017)

$$\dim_{\mathrm{loc}} \boldsymbol{\kappa}(x) = \dim(x)\,.$$

$$\mathrm{Dim}_{\mathrm{loc}} \boldsymbol{\kappa}(x) = \mathrm{Dim}(x)\,.$$

Very recently, J. Lutz and N. Lutz have generalized this theorem to all outer measures μ on $\mathbb{R}^n$ that are "locally optimal."

Some Classical Applications of the Point-to-Set Principle

N. Lutz and Stull 2020 : Improved lower bounds on the Hausdorff dimensions of generalized Furstenberg sets

N. Lutz 2021 : Extensions of Marstrand's fractal intersection formula for Hausodorff dimension from Borel sets to arbitrary sets

N. Lutz and Stull 2018 : Extension of Marstrand's projection theorem from analytic sets to arbitrary sets, provided that their Hausdorff dimension and packing dimensions coincide

T. Orponen 2021 : Classical proofs of two results in the preceding paper

Some Classical Applications of the Point-to-Set Principle continued

Slaman 2021 : If $V = L$, then the maximal thin co-analytic set has Hausdorff dimension 1

Stull 2022 : Further relaxation of the hypothesis of Marstrand's projection theorem

Stull 2024 : An improved bound on the Hausdorff dimensions of pinned distance sets

A simple application of the Point-to-Set Principle in more detail

A *Hamel basis* here is a basis of $\mathbb{R}$ as a vector space over $\mathbb{Q}$

Hamel 1905: Hamel bases exist and have the cardinality of the continuum.

Sierpinski 1920: Hamel bases have inner Lebesgue measure 0.

J. Lutz, Qi, & Yu 2024: For every $s \in [0, 1]$ there is a Hamel basis with Hausdorff dimension s.

Theorem (L, Qi, & Yu 2024)

For every $s \in [0,1]$ *there is a Hamel basis* B *of* $\mathbb{R}$ *over* $\mathbb{Q}$ *with* $\dim_{\mathrm{H}}(B) = s$.

Sketch of proof : let $s \in [0,1]$

By known methods, construct a Cantor-like set $C_s \subseteq [0,1]$ such that

- $\dim_{\mathrm{H}}(C_s) = s$;
- for all oracles $A \subseteq \mathbb{N}$ that compute s, the set $D^A = \{x \in C_s \mid \dim^A(x) = s\}$ has the cardinality of $\mathbb{R}$;
- Span$(C_s) = \mathbb{R}$.

Fix a wellordering

$$(x_\alpha | \alpha < 2^{\aleph_0})$$

of C_s and a wellordering

$$((A_\beta, y_\beta) | \beta < 2^{\aleph_0})$$

of the set

$$D = \{(A, y) \in P(\mathbb{N}) \times C_s | s \leq_T A \; and \; dim^A(y) = s\}$$

We define a sequence $(u_\gamma | \gamma < 2^{\aleph_0})$ by transfinite recursion, so that $B = \{u_\gamma | \gamma < 2^{\aleph_0}\}$ is the Hamel basis that we want.

We define a sequence $(u_\gamma | \gamma < 2^{\aleph_0})$ by transfinite recursion, so that $B = \{u_\gamma | \gamma < 2^{\aleph_0}\}$ is the Hamel basis that we want.

Given $\gamma < 2^{\aleph_0}$, let $B_\gamma = \{u_\delta | \delta < \gamma\}$. Write $\gamma = \xi + k$, where ξ is 0 or a limit ordinal and $k \in \mathbb{N}$. Call γ even/odd if k is even/odd.

1. If $\gamma = \xi + 2j$ is even, define

$$u_\gamma = y_\beta,$$

   where β is least such that $A_\beta = A_{\xi+j}$ and $y_\beta \notin span(B_\gamma)$.

2. If γ is odd, let

$$u_\gamma = x_\alpha$$

   be the first element of $C_s \backslash span(B_\gamma)$.

Routine: This recursion is well-defined, and B is a Hamel Basis

Clear: $B \subseteq C_s$, so $dim_H(b) \leq dim_H(C_s) = s$.

Proof that $\mathbf{dim_H(B) \geq s}$:

Let $s \in (0, 1]$, and let $A \subseteq \mathrm{N}$ with $s \leq_T A$.

We saw that

$$D^A = \{y \in C_s | dim^A(y) = s\}$$

has

$$|D^A| = 2^{\aleph_0},$$

so there exists ordinals $\gamma, \beta < 2^{\aleph_0}$ such that

$$A_\beta = A, y_\beta = u_\gamma \in B$$

Hence $dim^A(\mu_\gamma) = s$. Writing $\mu(A) = u_\gamma$ here, it follows that

$$
\begin{aligned}
dim_H(B) &\overset{\text{PSP}}{=} \min_{A \subseteq \mathbb{N}} \sup_{u \in B} dim^A(u) \\
&= \min_{A \subseteq \mathbb{N},\ s \leq_T A} \sup_{u \in B} dim^A(u) \\
&\geq \min_{A \subseteq \mathbb{N},\ s \leq_T A} dim^A(u(A)) \\
&= s \qquad \blacksquare
\end{aligned}
$$

Next goal

Understand the power and limitations of point-to-set principles for analytic concepts

Thank you!