Minimal covers in the Weihrauch degrees

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(joint work with J. Miller, Pauly, M. Soskova and Valenti)

A *mathematical problem* can be viewed as a statement of the form

$$\forall X\,(\varphi(X) \to \exists Y\,\psi(X, Y)),$$

where φ and ψ are formulas in the (two-sorted) language $\mathcal{L} = \{+, \cdot, <, 0, 1, \in\}$ using only number quantifiers.

Here, X is called an *instance*, and Y a *solution* of the problem.

Two standard examples are:

- *Weak König's Lemma*: X is an infinite binary tree by $\varphi(X)$, and Y is an infinite path through X by $\psi(X, Y)$;
- *Ramsey's Theorem for Pairs and 2 Colors*: X is a 2-coloring of unordered pairs of numbers by $\varphi(X)$, and Y is an infinite homogeneous set by $\psi(X, Y)$.

We consider mathematical problems from three angles: the proof-theoretic, the model-theoretic and the computability-theoretic one.

The proof-theoretic angle: *Reverse Mathematics*

We work over a weak base theory, usually RCA_0 (PA^- with Σ^0_1-Induction and Δ^0_1-Comprehension, essentially codifying computable mathematics), and measure the proof-theoretic strength of mathematical problems in the usual proof calculus.

E.g., one can show that Weak König's Lemma and Ramsey's Theorem for Pairs and 2 colors are independent over RCA_0. Ramsey's Theorem for Pairs with 2 colors and with 3 colors are equivalent, but strictly weaker than Ramsey's Theorem for Triples with 2 colors.

On the one hand, this approach is less restrictive: We can use assumptions repeatedly.
But our proof (thinking model-theoretically, i.e., semantically) has to work for any model of arithmetic, including non-standard models, which may not satisfy full (first-order) induction.
(E.g., the Infinite Pigeonhole Principle does not follow from RCA_0.)

The model-theoretic angle: $P \leq_\omega Q$

Instead of considering all models of our theory, we can only consider models with a standard first-order part (so-called *ω-models*, with an (often countable) second-order part $\mathcal{S} \subseteq \mathcal{P}(\omega)$).

We then work with semantic implication: A problem P is reducible to a problem Q if every model $(\omega, \mathcal{S})$ of Q is a model of P.

This approach has not been explored very much. (It is sometimes called the *ω-model reducibility* and denoted as $Q \models_\omega P$.)
It avoids "pesky" problems with induction.
E.g., the Infinite Pigeonhole Principle is just outright true (in ω-models of RCA_0).

The (less restrictive) computability-theoretic approach:

Call P *computably reducible* to Q ($P \leq_c Q$) if

- every P-instance X computes a Q-instance $\hat{X}$, and
- every Q-solution $\hat{Y}$ to this $\hat{X}$, together with X, computes a P-solution Y to X.

This approach is more restrictive: We can use assumptions only once but can argue computability-theoretically.
(If Y can be computed only from $\hat{Y}$ without using X, we write $P \leq_{sc} Q$.)

E.g., now Ramsey's Theorem for Pairs with 3 colors does not computably reduce to Ramsey's Theorem for Pairs with 2 colors.

The (more restrictive) computability-theoretic approach:
Weihrauch reducibility

We restrict the previous approach by requiring uniformity:
$P \leq_W Q$ if there are Turing functionals Φ and Ψ (the *forward* and the *backward* functionals) such that

- every P-instance X computes a Q-instance $\hat{X} = \Phi(X)$, and
- every Q-solution $\hat{Y}$ to this $\hat{X}$, together with X, uniformly computes a P-solution $Y = \Psi(\hat{Y} \oplus X)$ to X.

This is the most restrictive approach: We are allowed to query Q only once, and only uniformly so.
(If Y can be computed only from $\hat{Y}$ as $\Psi(\hat{Y})$, we write $P \leq_{sW} Q$.)

E.g., we have $\mathrm{DNR}_2 \leq_c \mathrm{DNR}_3$ but $\mathrm{DNR}_2 \not\leq_W \mathrm{DNR}_3$.

Much research about Weihrauch reducibility concerns applications, often via "representing" problems in other spaces via "names" in $\mathbb{N}^{\mathbb{N}}$.

However, we consider the *Weihrauch degrees* as a *degree structure*: So we change notation:

Consider problems as partial multi-valued functions $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$, mapping problems x satisfying $\varphi(x)$ to the set of all solutions y satisfying $\psi(x, y)$.

We denote the set of partial multi-valued functions $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ by $\mathcal{PF}$, and the quotient $(\mathcal{PF}/\equiv_W, \leq)$ (with the induced partial order) by $\mathcal{W}$.

Basic Facts about $\mathcal{W}$:

$\mathcal{W}$ is a partial order with least element $\mathbf{0} = \{\emptyset\}$.
Under AC, $\mathcal{W}$ has no greatest (or even maximal) element.

$\mathcal{W}$ has size $2^{\mathfrak{c}} = 2^{2^{\aleph_0}}$.
In fact, every Weihrauch degree $\neq \mathbf{0}$ has size $2^{\mathfrak{c}}$.

Every nontrivial lower cone in $\mathcal{W}$ has size $2^{\mathfrak{c}}$.

Every nontrivial maximal antichain in $\mathcal{W}$ must be uncountable.
There is a maximal antichain of size $2^{\mathfrak{c}}$, but nothing more is known.

Every well-ordered ascending chain in $\mathcal{W}$ of countable cofinality has an upper bound.
For every $\kappa \leq \mathfrak{c}$ of uncountable cofinality, there is an ascending chain in $\mathcal{W}$ of type κ without upper bound.
(This is open for $\mathfrak{c} < \kappa \leq 2^{\mathfrak{c}}$.)

Quite a few natural operations on $\mathcal{PF}$ have been defined, some of which are degree-theoretic, and some of which are not.

The following operations of meet and join make $\mathcal{W}$ into a distributive lattice:

$$f \sqcup g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}, \quad (f \sqcup g)(i,x) = \begin{cases} \{0\} \times f(x), & \text{if } i = 0, \\ \{1\} \times g(x), & \text{if } i = 1; \end{cases}$$

$$f \sqcap g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}, \quad (f \sqcap g)(x,y) = (\{0\} \times f(x)) \cup (\{1\} \times g(y)).$$

The next "natural" degree-theoretic question concerns the (un)decidability and complexity of the first-order theory of $\mathcal{W}$.

The Weihrauch degree $\mathbf{1} = \deg(\mathrm{id})$ of the identity function

$$\mathrm{id} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, x \mapsto x$$

plays a special role as we will now explore.

The Lattice of the Medvedev Degrees:

A *mass problem* is a subset $\mathcal{A} \subseteq \mathbb{N}^\mathbb{N}$.

A mass problem $\mathcal{A}$ is *Medvedev reducible* to a mass problem $\mathcal{B}$ ($\mathcal{A} \leq_M \mathcal{B}$) if there is a Turing functional Φ such that $\Phi(\mathcal{B}) \subseteq \mathcal{A}$. (So, in particular, $\Phi(x)$ is a total function for all $x \in \mathcal{B}$.)
Denote the quotient $(\mathcal{P}(\mathbb{N}^\mathbb{N})/\equiv_M, \leq)$ of *Medvedev degrees* by $\mathcal{M}$.

We next define, for each $\mathcal{A} \subseteq \mathbb{N}^\mathbb{N}$, the function $d_\mathcal{A} :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ mapping each $x \in \mathcal{A}$ to 0^ω. (Note $d_\mathcal{A} \equiv_W \mathrm{id} \restriction \mathcal{A}$.)

Then the map $d : \mathcal{P}(\mathbb{N}^\mathbb{N}) \to \mathcal{PF}$, $\mathcal{A} \mapsto d_\mathcal{A}$ induces an embedding of $\mathcal{M}^{op}$ (the Medvedev degrees under the *reverse* ordering) into $\mathcal{W}$ (by Higuchi/Kihara 2013, following Brattka/Gherardi 2011). This embedding is *onto* the cone $\mathcal{W}(\leq \mathbf{1})$ in the Weihrauch degrees below $\deg_W(\mathrm{id})$.

So note $\mathcal{M}^{op} \cong \mathcal{W}(\leq \mathbf{1}) = \{\deg_W(\mathrm{id} \restriction \mathcal{A}) \mid \mathcal{A} \subseteq \mathcal{P}(\mathbb{N}^\mathbb{N})\}$.

Question (Pauly 2020)

Is $\mathbf{1} = \deg_W(\text{id})$ definable in $(\mathcal{W}, \leq)$?

Theorem (Lempp, J. Miller, Pauly, M. Soskova, Valenti)

The degree $\mathbf{1}$ is definable in $(\mathcal{W}, \leq)$ in two ways:

1. $\mathbf{1}$ is the greatest degree that is a strong minimal cover in $\mathcal{W}$.
2. $\mathbf{1}$ is the least degree such that the cone above it is dense.

Theorem (Lewis-Pye, Nies, Sorbi 2009, Shafer 2011)

The first-order theory of $(\mathcal{M}, \leq)$ is as complicated as third-order arithmetic.

Corollary (Lempp, J. Miller, Pauly, M. Soskova, Valenti)

The first-order theory of $(\mathcal{W}, \leq)$ (and of $(\mathcal{W}(\leq \mathbf{1}), \leq)$) is as complicated as third-order arithmetic.

Proof Sketch (1):

For $x \in \mathbb{N}^{\mathbb{N}}$, let $\{x\}^+ = \{(e)^\frown y \mid \Phi_e(y) = x \text{ and } y \not\leq_T x\}$.

Theorem (Dyment 1976)

In the lattice of the Medvedev degrees $(\mathcal{M}, \leq, \wedge, \vee)$:

- $\mathcal{B}$ is a *minimal cover* of $\mathcal{A}$ iff there is $x \in \mathcal{A}$ with $\mathcal{A} \equiv_M \mathcal{B} \wedge \{x\}$ and $\mathcal{B} \wedge \{x\}^+ \equiv_M \mathcal{B}$.
- The *strong minimal covers* are precisely of the form $(\deg_M(\{x\}), \deg_M(\{x\}^+))$ for any $x \in \mathbb{N}^{\mathbb{N}}$.

So being the Medvedev degree of a singleton (i.e., being a *degree of solvability*) is definable in $\mathcal{M}$.

Corollary

$\mathbf{1} = \deg_W(\text{id})$ is a strong minimal cover of $\deg_W(\text{id} \restriction \mathrm{NREC})$, where $\mathrm{NREC} = \deg_M(\{0^\omega\}^+) = \deg_M(\{x \in \mathbb{N}^{\mathbb{N}} \mid x >_T 0^\omega\})$.

Theorem (Lempp, J. Miller, Pauly, M. Soskova, Valenti)

In the Weihrauch degrees $(\mathcal{W}, \leq)$:

- $\deg_W(g)$ is a *minimal cover* of $\deg_W(f)$ iff $g \equiv_W f \sqcup id \restriction \{x\}$ for some $x \in \mathbb{N}^{\mathbb{N}}$ with $\mathrm{dom}(f) \not\leq_M \{x\}$ and $\mathrm{dom}(f) \leq_M \{x\}^+$.
- $\deg_W(g)$ is a *strong minimal cover* of $\deg_W(f)$ iff there is $x \in \mathbb{N}^{\mathbb{N}}$ with $g \equiv_W \mathrm{id} \restriction \{x\}$ and $f \equiv_W \mathrm{id} \restriction \{x\}^+$.

In particular, $\deg_W(\mathrm{id})$ is the greatest strong minimal cover in $\mathcal{W}$, and every Weihrauch degree has at most one strong minimal cover.

Our proof critically relies on the following

Lemma

- If $\deg_W(g)$ is a minimal cover of $\deg_W(f)$, then there is h with $|\mathrm{dom}(h)| = 1$ such that $g \equiv_W f \sqcup h$.
- If $\deg_W(g)$ is a strong minimal cover of $\deg_W(f)$, then there is $h \equiv_W g$ with $|\mathrm{dom}(h)| = 1$.

Proof of "If $\deg_W(g)$ is a minimal cover of $\deg_W(f)$, then there is h with $|\text{dom}(h)| = 1$ such that $g \equiv_W f \sqcup h$."

Construct $\xi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ as $\xi = \bigcup_{s \in \omega} \xi_s$ for finite functions ξ_s.

Define $G_\xi(x, \xi(x)) = g(x)$.

Then $G_\xi \leq_W g$ for all ξ. We *try* to ensure $f <_W f \sqcup G_\xi <_W g$ by letting ξ "scramble" the domain of g.

At odd stages, we try to ensure $G_\xi \not\leq_W f$ via the pair (Φ_e, Φ_i).

At even stages we try to ensure $g \not\leq_W f \sqcup G_\xi$ via the pair (Φ_e, Φ_i).

So this construction has to start failing at some finite stage s with some ξ_s.
This gives $g \equiv_W f \sqcup G_{\xi_s}$ for a finite function G_{ξ_s}.
But $G_{\xi_s} \equiv_W \bigsqcup_{i_n} h_i$ for functions h_i with $|\text{dom}(h_i)| = 1$.
Since $\deg_W(g)$ is a minimal cover of $\deg_W(f)$, we have $g \equiv_W f \sqcup h_j$ for some $j < n$.

Proof Sketch (2):

We rely on the following

Lemma

The following are equivalent for $f \in \mathcal{PF}$:

- id $\not\leq_W f$;
- There are $g, h \in \mathcal{PF}$ such that $f \leq_W g <_W h$ and $\deg_W(h)$ is a minimal cover of $\deg_W(g)$.

Thus, in particular, the Weihrauch degrees $\geq \mathbf{1}$ are dense, and $\mathbf{1}$ is least such.

Thank you!