Isomorphism problems and learning

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UW Logic Seminar (online)

March 12, 2024

In this talk all the families $\mathfrak{K}$ of structures we consider are *countable* and every structure has domain $\mathbb{N}$ and a *relational signature*.

- a structure $\mathcal{A}$ is identified via its *atomic diagram*, i.e. the collection of atomic formulas that are true in $\mathcal{A}$. Up to a suitable Gödel numbering of formulas, $\mathcal{A}$ can be identified with some $p \in 2^{\mathbb{N}}$.
- Every $\mathcal{A}$ is assigned a *code* $\ulcorner \mathcal{A} \urcorner \in \mathbb{N}$: let $\mathrm{HS}(\mathfrak{K}) := \{\ulcorner \mathcal{A} \urcorner : \mathcal{A} \in \mathfrak{K}\} \cup \{?\}$.

Two characters, a *learner* **M** and an *opponent*, both having access to $\mathfrak{K}$.

- Let $\mathrm{LD}(\mathfrak{K}) := \bigcup_{\mathcal{A} \in \mathfrak{K}} \{\mathcal{S} : \mathcal{S} \cong \mathcal{A}\}$.
  - The opponent picks some $\mathcal{S} \in \mathrm{LD}(\mathfrak{K})$ (without revealing its choice to **M**).
- **M** sees larger and larger finite pieces of the atomic diagram of $\mathcal{S}$.
  - **M** is formalized as a function from $2^{<\mathbb{N}}$ to $\mathrm{HS}(\mathfrak{K})$.

$\mathfrak{K}$ is **Ex**-*learnable* if there exists a learner **M** such that for every $\mathcal{S} \in \mathrm{LD}(\mathfrak{K})$,

$$\lim_{n \to \infty} \mathbf{M}(\mathcal{S} \restriction_n) = \ulcorner \mathcal{A} \urcorner \iff \mathcal{S} \cong \mathcal{A}\,.$$

where $\mathcal{S} \restriction_n$ is the restriction of $\mathcal{S}$ to the domain $\{0, \dots, n\}$.

EXAMPLE: $\mathfrak{K} = \{\omega, \omega^*\}$

- ω is the linear order having order type the natural numbers;
- ω^* is the linear order having order type the negative integers.

Is $\mathfrak{K}$ **Ex**-learnable? Given $\mathcal{S} \in \mathrm{LD}(\mathfrak{K})$, let

$$min_s := \min\{n : n \in \mathcal{S} \restriction_s\} \text{ and } max_s := \max\{n : n \in \mathcal{S} \restriction_s\},$$

$$c(min_s) := |\{t \leq s : min_s = min_{s-t}\}| \text{ and } c(max_s) := |\{t \leq s : max_s = max_{s-t}\}|.$$

Informally, $c(min_s) > c(max_s)$ can be interpreted as "is plausible that $\mathcal{S} \restriction_s$ will be extended to a copy of ω" (similarly, for $c(min_s) < c(max_s)$ and ω^*). Hence we define a learner **M** so that

$$\mathbf{M}(\mathcal{S} \restriction_s) = \begin{cases} \ulcorner \omega \urcorner & \text{if } c(min_s) \geq c(max_s), \\ \ulcorner \omega^* \urcorner & \text{if } c(min_s) < c(max_s). \end{cases}$$

M **Ex**-learns $\mathfrak{K}$.

A "DEFECT" OF THE FRAMWORK

So far, we can "only" say whether a family $\mathfrak{K}$ is **Ex**-learnable or not.
Together with Bazhenov and San Mauro we proposed a framework to classify the nonlearnability of a family borrowing ideas from *descriptive set theory*.

Why DST?

- It focuses on classification problems (in particular, isomorphism problems), through reducibility between equivalence relations, and isomorphism has a pivotal role in our paradigm: the nonlearnability of a family $\mathfrak{K}$ is rooted in the complexity of the isomorphism relation associated with $\mathfrak{K}$.
- It gives us a framework to study the complexity of equivalence relations on topological spaces.

An equivalence relation E is reducible to F it there is a (nice) function $\Gamma : X \to Y$ such that $x \; E \; x' \iff \Gamma(x) \; F \; \Gamma(x')$.

	DST	vs	Our study
focus	*large* collections of countable structures		*small* (countable) families
reduction	Borel		continuous

Our reductions are continuous

Continuous reductions "mimic" the behavior of the learner: given a finite portion of the structure, without knowing how it will be extended, it must output a conjecture.
This choice is further supported by the following result.
Given $p, q \in 2^{\mathbb{N}}$, let $p \; E_0 \; q :\iff (\exists n)(\forall m \geq n)(p(m) = q(m))$.

Theorem (Bazhenov, C., San Mauro)

Let $\mathfrak{K}$ be a family of structures. Then $\mathfrak{K}$ is **Ex**-*learnable iff $\mathrm{LD}(\mathfrak{K})$ continuously reduces to E_0, i.e., there is a continuous function Γ s.t. for all $\mathcal{A}, \mathcal{B} \in \mathrm{LD}(\mathfrak{K})$*

$$\mathcal{A} \cong \mathcal{B} \iff \Gamma(\mathcal{A}) \; E_0 \; \Gamma(\mathcal{B}).$$

Replacing E_0 with other equivalence relations one "unlocks" the promised hierarchy.

Definition (Bazhenov, C., San Mauro)

$\mathfrak{K}$ is *E-learnable* if there is a function Γ that continuously reduces $\mathrm{LD}(\mathfrak{K})$ to E.
I.e. for every $\mathcal{A}, \mathcal{B} \in \mathrm{LD}(\mathfrak{K})$, $\mathcal{A} \cong \mathcal{B} \iff \Gamma(\mathcal{A}) \; E \; \Gamma(\mathcal{B})$.

E-learnability

Definition (Bazhenov, C., San Mauro)

$\mathfrak{K}$ is *E-learnable* if there is a function Γ that continuously reduces $\mathrm{LD}(\mathfrak{K})$ to E. I.e. for every $\mathcal{A}, \mathcal{B} \in \mathrm{LD}(\mathfrak{K})$, $\mathcal{A} \cong \mathcal{B} \iff \Gamma(\mathcal{A})\ E\ \Gamma(\mathcal{B})$.

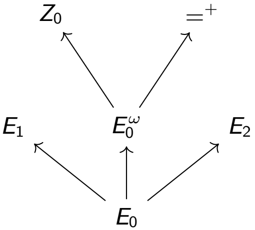
Together with Bazhenov and San Mauro we introduced the notion of *learn reducibility*. Namely,

- E is learn reducible to F ($E \leq_{\mathrm{Learn}} F$), if every E-learnable family of structures is also F-learnable.
- *Finitary learn reducibility* ($E \leq^{<\aleph_0}_{\mathrm{Learn}} F$) is defined similarly but restricting to finite families.

The two notions of reducibility behave quite differently and now we list some of the results we have in this direction.

When "stronger" E's do not help

This is the picture under continuous reducibility.
Here $=$ denotes the equality on $2^{\mathbb{N}}$.



N.B.: if E cont. red. to F, then $E \leq_{\text{Learn}} F$.

Theorem (Bazhenov, C., San Mauro)

$E_0 \equiv_{\text{Learn}} E_1 \equiv_{\text{Learn}} E_2$.

E_0^ω AND $=^+$

Given $p \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$, let $p^{[m]}$ be the *m-th column of p*, i.e., $p^{[m]} := p(m, \cdot)$.

Definition

Let E be an equivalence relation on a standard Borel space X.
The *power* of E is the equivalence relation E^ω on $X^\mathbb{N}$ defined by

$$p \; E^\omega \; q :\Longleftrightarrow \; (\forall n)(p^{[n]} \; E \; q^{[n]}).$$

The *FS jump* of E, is the equivalence relation E^+ on $X^\mathbb{N}$ defined by

$$p \; E^+ \; q :\Longleftrightarrow \; \{[p^{[n]}]_E : n \in \mathbb{N}\} = \{[q^{[n]}]_E : n \in \mathbb{N}\}.$$

E.g., $p \; =^+ \; q \iff \{p^{[n]} : n \in \mathbb{N}\} = \{q^{[n]} : n \in \mathbb{N}\}$.

Theorem (Bazhenov, C., San Mauro)

$E_0 \equiv^{<\aleph_0}_{\mathrm{Learn}} E_0^\omega$ *but* $E_0 <_{\mathrm{Learn}} E_0^\omega$.
$E_0 <^{<\aleph_0}_{\mathrm{Learn}} =^+$.

The strength of $=^+$ lies in

- the opportunity of adding "garbage" columns at any stage,
- not caring about the multiplicity of the columns.

Characterizing learning criteria

A characterization for E_0

Theorem (Bazhenov, Fokina, San Mauro)

Let $\mathfrak{K} = \{\mathcal{A}_i : i \in \mathbb{N}\}$ be a family of structures: $\mathfrak{K}$ is **Ex***-learnable iff there is a sequence $\{\psi_i : i \in \mathbb{N}\}$ of Σ_2^{inf} sentences s.t. for every i, j, $\mathcal{A}_j \models \psi_i \iff i = j$*

As an application of this theorem, recall $\{\omega, \omega^*\}$ and let

$$\psi_\omega := (\exists n)(\forall m)(n < m) \text{ and } \psi_{\omega^*} := (\exists n)(\forall m)(n > m).$$

Clearly, $\omega \models \psi_\omega$ and $\omega \not\models \psi_{\omega^*}$ (similarly for ω^* swapping ψ_ω and ψ^*_ω).

To show the non-learnability of a family, consider $\{\omega, \zeta\}$.
Notice that for any Σ_2^{inf} formula φ, if $\zeta \models \varphi$ then also $\omega \models \varphi$ and indeed, $\{\omega, \zeta\}$ is not **Ex**-learnable.

For a given n and a structure $\mathcal{A}$, let $Th_{\Sigma_n^{inf}}(\mathcal{A}) := \{\varphi \in \Sigma_n^{inf} : \mathcal{A} \models \varphi\}$. Let $\mathfrak{K} = \{\mathcal{A}_i : i \in \mathbb{N}\}$ be a family of structures. Then,

- $\mathfrak{K}$ is a *Σ_n^{inf}-strong antichain* if there are Σ_n^{inf} formulas $\{\varphi_i : i \in \mathbb{N}\}$ so that $\mathcal{A}_i \models \varphi_j \Leftrightarrow i = j$.
  - Hence, we can rephrase the theorem above as "$\mathfrak{K}$ is E_0-learnable iff $\mathfrak{K}$ is a Σ_2^{inf}-strong antichain.".
- $\mathfrak{K}$ is a *Σ_n^{inf}-antichain*, if any two structures in $\mathfrak{K}$ are incomparable with respect to $\subseteq_{Th_{\Sigma_n^{inf}}}$;
- $\mathfrak{K}$ is a *Σ_n^{inf}-poset* if $\mathfrak{K}$ ordered by $\subseteq_{Th_{\Sigma_n^{inf}}}$ is a poset.

Observation: if $\mathfrak{K}$ is finite, then the notions of Σ_n^{inf}-strong antichain and Σ_n^{inf}-antichain coincide.

Σ^{inf}_1-(STRONG) ANTICHAINS

Fin-learnability is **Ex**-learnability where the learner never changes its mind.
We denote by $=$ equality on $2^{\mathbb{N}}$ and by $=_{\mathbb{N}}$ equality on $\mathbb{N}$.
It is easy to see that **Fin**-learnability corresponds to $=_{\mathbb{N}}$-learnability.

Theorem (Bazhenov, C., Jain, San Mauro, Stephan)

$\mathfrak{K}$ is $=_{\mathbb{N}}$-learnable iff $\mathfrak{K}$ is a Σ^{inf}_1-strong antichain.
$\mathfrak{K}$ is $=$-learnable iff $\mathfrak{K}$ is a Σ^{inf}_1-antichain.

Proposition

Since for finite families, Σ^{inf}_1-strong antichains and Σ^{inf}_1-antichains coincide, $=_{\mathbb{N}} \equiv^{<\aleph_0}_{\mathrm{Learn}} =$. On the other hand, the family consisting of all finite cyclic graphs and the infinite ray is $=$-learnable but not $=_{\mathbb{N}}$-learnable ($=_{\mathbb{N}} <_{\mathrm{Learn}} =$).

Let α-learnability be **Ex**-learnability where at most α-mind changes are allowed.

Theorem (Bazhenov, C., San Mauro)

Let $\mathfrak{K}$ be an $=$-learnable family: $\mathfrak{K}$ is α-learnable iff $\mathrm{range}(\Gamma)^{1+\alpha} = \emptyset$. *In particular, $\mathfrak{K}$ is $=_{\mathbb{N}}$-learnable iff* $\mathrm{range}(\Gamma)$ *has no limit points.*

Σ_1^{inf}-POSETS

$p =^+_{\mathbb{N}} q :\iff \{p(n) : n \in \mathbb{N}\} = \{q(n) : n \in \mathbb{N}\}$.

Theorem (Bazhenov, C., Jain, San Mauro, Stephan)

$\mathfrak{K}$ is $=^+_{\mathbb{N}}$-learnable iff $\mathfrak{K}$ is a Σ_1^{inf}-poset.

Proof idea for $\Rightarrow$: suppose that $\mathfrak{K}$ is $=^+_{\mathbb{N}}$-learnable by Γ and that there exists $\mathcal{A}_i \ncong \mathcal{A}_j \in \mathfrak{K}$ s.t. $Th_{\Sigma_1^{inf}}(\mathcal{A}_i) = Th_{\Sigma_1^{inf}}(\mathcal{A}_j)$. In order to be a reduction from $\mathrm{LD}(\mathfrak{K})$ to $=^+_{\mathbb{N}}$ there must be some $n \in \Gamma(\mathcal{A}_i) \setminus \Gamma(\mathcal{A}_j)$. Hence, if $\mathcal{S} \cong \mathcal{A}_i$, there exists a stage s such that $\Gamma(\mathcal{S})(s) = n$. On the other hand, since $Th_{\Sigma_1^{inf}}(\mathcal{A}_i) = Th_{\Sigma_1^{inf}}(\mathcal{A}_j)$, at any finite stage $\mathcal{S}$ may be extended to a copy of $\mathcal{A}_i$ or $\mathcal{A}_j$, (*e.g., $\{\omega, \omega^*\}$ is not $=^+_{\mathbb{N}}$-learnable*).
Proof idea for $\Leftarrow$: for every $\mathcal{A}_i, \mathcal{A}_j \in \mathfrak{K}$, let φ_{ij} be the Σ_1^{inf} formula such that $\mathcal{A}_i \models \varphi_{ij}$ and $\mathcal{A}_j \not\models \varphi_{ij}$ and assign a code $\langle i, j \rangle$ to it. Then given $\mathcal{S} \in \mathrm{LD}(\mathfrak{K})$, define a leaner outputting $\langle i, j \rangle$ at stage s if $\mathcal{S} \upharpoonright_s \models \varphi_{ij}$.

Σ_2^{inf}-(STRONG) ANTICHAINS

Theorem (Bazhenov, Fokina, San Mauro)

$\mathfrak{K}$ is E_0-learnable iff $\mathfrak{K}$ is a Σ_2^{inf}-strong antichain.

The following paradigm is well-studied in classical inductive inference.
A family of structures $\mathfrak{K}$ is **PL**-*learnable* if there exists a learner **M** such that for every $\mathcal{S} \in \mathrm{LD}(\mathfrak{K})$,

$$|\{n : \mathbf{M}(\mathcal{S} \restriction_n) = \ulcorner \mathcal{A} \urcorner\}| = \infty \iff \mathcal{A} \cong \mathcal{S}\,.$$

Theorem (Bazhenov, C., Jain, San Mauro, Stephan)

*$\mathfrak{K}$ is **PL**-learnable iff $\mathfrak{K}$ is a Σ_2^{inf}-antichain.*

Proposition

*$E_0 \equiv^{<\aleph_0}_{\mathrm{Learn}}$ **PL**. On the other hand, the family consisting of all finite linear orders and ω is **PL**-learnable but not **Ex**-learnable ($E_0 <_{\mathrm{Learn}}$ **PL**).*

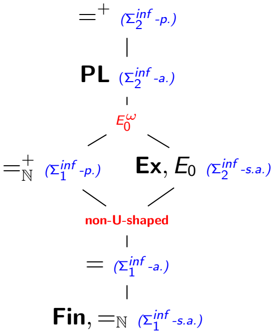
Σ_2^{inf}-POSETS

$$p =^+ q :\iff \{p^{[n]} : n \in \mathbb{N}\} = \{q^{[n]} : n \in \mathbb{N}\}.$$

Theorem (C., Marcone, San Mauro)

$\mathfrak{K}$ is $=^+$-learnable iff $\mathfrak{K}$ is a Σ_2^{inf}-poset.

We will say something more in the next slides about the Σ_n^{inf}-poset case. Let's summarize what we have. Is there something in between?



- **non-U-shaped**-learnability: after the first time the learner outputs the correct hypothesis, it cannot change its mind.

Work in progress!

We have that $=^+_{\mathbb{N}}$ and $=^+$ characterize respectively the Σ^{inf}_1- and Σ^{inf}_2- posets. Notice that, $= \sim_B =^+_{\mathbb{N}}$ and hence $=^+ \sim_B =^{++}_{\mathbb{N}}$: this is not true in our context.

"Theorem" (C., Marcone, San Mauro)

Let $\mathfrak{K}$ be a family of structures.

- *$\mathfrak{K}$ is $=^{(n+1)+}_{\mathbb{N}}$-learnable iff $\mathfrak{K}$ is a Σ^{inf}_{2n+1}-poset.*
- *$\mathfrak{K}$ is $=^{(n+1)+}$-learnable iff $\mathfrak{K}$ is a Σ^{inf}_{2n+2}-poset.*

N.B.: some details of the proof need to be checked, but we describe its two key steps. We only discuss the second item, the first one being similar.
The proof has essentially two main steps:

- show the theorem for families of size 2;
- show that if a family $\mathfrak{K}$ fails to be $=^{n+}$-learnable this difficulty lies into a pair of structures in the family.

Step 1

$\{\mathcal{A},\mathcal{B}\}$ is $=^{(n+1)+}$-learnable iff $Th_{\Sigma^{inf}_{2n+2}}(\mathcal{A}) \neq Th_{\Sigma^{inf}_{2n+2}}(\mathcal{B})$.
($\Longleftarrow$, by induction)

- (Base case, $=^+$). Let φ be the Σ^{inf}_2 formula satisfied by $\mathcal{A}$ and not by $\mathcal{B}$. The formula φ is of the form $\exists i \bigwedge\!\!\!\bigwedge_j \forall t \psi_{ij}(i,t)$ where ψ_{ij} is a Δ^{inf}_0 formula. We define a reduction Γ from $\mathrm{LD}(\{\mathcal{A},\mathcal{B}\})$ to $=^+$ as follows. For any i,
  - in the odd columns, we put garbage, i.e., $\Gamma(\mathcal{S})^{[2i+1]} = 0^i 1^{\mathbb{N}}$;
  - in the even ones we let $\Gamma(\mathcal{S})^{[2i]} = 0^{\mathbb{N}}$ if $\mathcal{S} \models \varphi$, and $0^k 1^{\mathbb{N}}$ where $k = \min\{j : \mathcal{S} \not\models \psi_{ij}(i,t)\}$, otherwise.

  We have that Γ is a "*nice*" reduction from $\mathrm{LD}(\{\mathcal{A},\mathcal{B}\})$ to $=^+$. Indeed,
  - $\mathcal{S} \cong \mathcal{A} \implies \{\Gamma(\mathcal{S})^{[m]} : m \in \mathbb{N}\} = \{0^i 1^{\mathbb{N}} : i \in \mathbb{N}\} \cup \{0^{\mathbb{N}}\}$;
  - $\mathcal{S} \ncong \mathcal{B} \implies \{\Gamma(\mathcal{S})^{[m]} : m \in \mathbb{N}\} = \{0^i 1^{\mathbb{N}} : i \in \mathbb{N}\}$.

  To avoid too many notations we will just sketch how to go from Σ^{inf}_2 to Σ^{inf}_4, showing that $\mathrm{LD}(\{\mathcal{A},\mathcal{B}\})$ continously reduces to $=^{++}$.
  Let $F, I \in (2^{\mathbb{N}})^{\mathbb{N}}$ be s.t. $F^{[0]} := 0^{\mathbb{N}}$ and $F^{[i+1]} := 0^i 1^{\mathbb{N}}$ and let I be s.t. $I^{[i]} := F^{[i+1]}$.

FROM Σ_2^{inf} TO Σ_4^{inf}

From the base case, we can assume that, given a Σ_2^{inf} formula φ and a structure $\mathcal{S}$, there is a continuous Φ s.t.

$$(*)\ \textit{if } \mathcal{S} \cong \varphi \text{ then } \Phi(\mathcal{S}) =^{+} F \text{ and } \Phi(\mathcal{S}) =^{+} I \text{ otherwise.}$$

Let φ be the Σ_4^{inf} formula satisfied by $\mathcal{A}$ and not by $\mathcal{B}$. The formula φ is of the form $\exists i \bigwedge\!\!\!\bigwedge_j \forall t \psi_{ij}(i, t)$ where ψ_{ij} is a Σ_2^{inf} formula and from the base case, I have an associated Φ_{ij} satisfying (*).

We want to define a reduction Γ from $\mathrm{LD}(\{\mathcal{A}, \mathcal{B}\})$ to $=^{++}$. Instead of doing so, we show a reduction from $\mathrm{LD}(\{\mathcal{A}, \mathcal{B}\})$ to $=^{+\omega+}$ and then we use a lemma that in particular implies that:

$$=^{++} \sim_c =^{+\omega+}.$$

FROM $n = 0$ TO $n = 1$ CTD.

We fill the columns of Γ as follows. As before, the odd columns are "garbage" ones, i.e. we just let ${\Gamma^{[2i+1]}}^{[j]} =^+ F$ if $j \leq i$, and ${\Gamma^{[2i+1]}}^{[j]} =^+ I$ otherwise.

informally, $\Gamma^{[2i+1]}$ is the higher dimension counterpart of $0^i 1^{\mathbb{N}}$.

In the even columns, we add the $\Phi_{ij}(\mathcal{S})$'s via a *permission system*. For any i, let $i_j := \min\{j : \mathcal{S} \not\models \psi_{ij}(i,t)\}$ (if exists). By definition, for every $j < i_j$, $\Phi_{ij}(\mathcal{S}) =^+ F$. Our permission system guarantees the following:

- if i_j exists, then ${\Gamma^{[2i]}}^{[j]} =^+ \begin{cases} F & \text{if } j < i_j, \\ I & \text{otherwise.} \end{cases}$
- otherwise, ${\Gamma^{[2i]}}^{[j]} =^+ F$ for every j.

Informally, if i_j does not exist, $\Gamma^{[2i]}$ is the higher dimension counterpart of $0^{\mathbb{N}}$.

- if $\mathcal{S} \cong \mathcal{A}$ then $\{\Gamma(\mathcal{S})^{[m]} : m \in \mathbb{N}\} = \{G^{[n]} : n \in \mathbb{N}\}$
- if $\mathcal{S} \ncong \mathcal{A}$ then $\{\Gamma(\mathcal{S})^{[m]} : m \in \mathbb{N}\} = \{J^{[n]} : n \in \mathbb{N}\}$

where $G, J \in 2^{\mathbb{N}^{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}}}$ are the higher dimension counterparts of F and I. That is, for every n,

- ${G^{[0]}}^{[i]} =^+ F$ for every i and ${G^{[n+1]}}^{[i]} =^+ F$ if $i < n$ and ${G^{[n+1]}}^{[i]} =^+ I$ otherwise;
- $J^{[n]} = G^{[n+1]}$.

This concludes the sketch of the proof for the Σ^{inf}_4 case. Notice that we can again assume to have a *nice* reduction, say Φ, satisfying an analogous condition (*) for this level. That is, given a structure $\mathcal{S}$ and Σ^{inf}_4 formula,

$$(*) \text{ if } \mathcal{S} \cong \varphi \text{ then } \Phi(\mathcal{S}) =^{+\omega+} G \text{ and } \Phi(\mathcal{S}) =^{+\omega+} J \text{ otherwise.}$$

A recap. We are in **Step 1**, $\big(\{\mathcal{A}, \mathcal{B}\}$ is $=^{(n+1)+}$-learnable iff

$Th_{\Sigma^{inf}_{2n+2}}(\mathcal{A}) \neq Th_{\Sigma^{inf}_{2n+2}}(\mathcal{B})\big)$ and we have sketched the $\Longleftarrow$ direction.
For the opposite direction, the claim should follow from classical results involving Turing computable embeddings and the pullback theorem.

So far we have just discussed the proof strategy for pairs of structures. What about the full theorem, i.e., what about infinite families? For this, we go to **Step 2**, that is proving that $=^{n+}_{\mathbb{N}}$ and $=^{n+}$ are *supercompact*.

Definition (C., Marcone, San Mauro)

We say that an equivalence relation E is *supercompact for learning* if, for any family of structures $\mathfrak{K}$, $\mathfrak{K}$ is E-learnable iff for every $\mathcal{A} \not\cong \mathcal{B} \in \mathfrak{K}$, $\{\mathcal{A}, \mathcal{B}\}$ is E-learnable.

Theorem (C., Marcone, San Mauro)

For any $n > 0$, $=^{n+}_{\mathbb{N}}$ and $=^{n+}$ are supercompact for learning.

Proof idea: By what we have discussed previously, we can assume that our reductions are *nice*. Then, given $\mathfrak{K}$ consider all reductions for pairs of structures in $\mathfrak{K}$ and (with some care) combine all of them.

Once we have supercomactness, the characterizations of $=^{n+}_{\mathbb{N}}$ and $=^{n+}$ in terms of Σ^{inf}_{2n+1}- and Σ^{inf}_{2n+2}-posets follows.

Supercompactness ctd.

Supercompactness for learning indicates that the non-learnability of a family is rooted in a pair of structures, and not in the infinite size of the family.
Notice that this easily fails for other E-learnabilities: for example, the family consisting of all finite linear orders and ω is not E_0-learnable, but any pair of structures in such a family is. In other words, we have just proved that

Corollary (C., Marcone, San Mauro)

E_0 is not supercompact for learning.

Future work and partial results

- Filling the missing details of the proof, also for the transfinite levels;
- characterize other learning criteria (we already have it for E_0^ω);
- study more in detail the learning hierarchy.

Thanks!

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