A Jump in the Weihrauch degrees

Steffen Lempp

University of Wisconsin-Madison

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(joint work with Andrews, Marcone, J. Miller and Valenti)

A *mathematical problem* can be viewed as a statement of the form

$$\forall X\,(\varphi(X) \to \exists Y\,\psi(X, Y)),$$

where φ and ψ are formulas in the (two-sorted) language $\mathcal{L} = \{+, \cdot, <, 0, 1, \in\}$ using only number quantifiers.

Here, X is called an *instance*, and Y a *solution* of the problem.

Two standard examples are:

- *Weak König's Lemma*: X is an infinite binary tree by $\varphi(X)$, and Y is an infinite path through X by $\psi(X, Y)$;
- *Ramsey's Theorem for Pairs and 2 Colors*: X is a 2-coloring of unordered pairs of numbers by $\varphi(X)$, and Y is an infinite homogeneous set by $\psi(X, Y)$.

This talk focuses on the Weihrauch degrees as a degree structure, rather than applications to measure the computational complexity of "ordinary" mathematical problems.
So will restrict ourselves to $X, Y \in \mathbb{N}^{\mathbb{N}}$ for the rest of the talk.

Consider problems as partial multi-valued functions $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$, mapping problems x satisfying $\varphi(x)$ to the set of all solutions y satisfying $\psi(x, y)$.

Definition (Weihrauch reducibility)

Let $f, g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ be partial multi-valued functions.
Then f is *Weihrauch reducible to g* (written $f \leq_\mathrm{W} g$) if the problem of computing f can be computably and uniformly solved by using for each instance of f a single computation of g in the following sense:
There are Turing functionals Φ and Ψ such that

1. $\Phi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ computes from every $p \in \mathrm{dom}(f)$ some $q \in \mathrm{dom}(g)$; and
2. $\Psi :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ computes from p and every $r \in g(q)$ some element of $f(p)$.

Basic Facts:

Let $\mathcal{W} = (\mathcal{PF}/\equiv_W, \leq)$ (for the set $\mathcal{PF}$ of partial multi-valued functions $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$), under the partial order induced by $\leq_W$.

$\mathcal{W}$ is a partial order with least element $\mathbf{0} = \{\emptyset\}$.
Under AC, $\mathcal{W}$ has no greatest (or even maximal) element.

$\mathcal{W}$ has size $2^\mathfrak{c} = 2^{2^{\aleph_0}}$.
In fact, every Weihrauch degree $\neq \mathbf{0}$ has size $2^\mathfrak{c}$.

Every nontrivial lower cone in $\mathcal{W}$ has size $2^\mathfrak{c}$.

Every nontrivial maximal antichain in $\mathcal{W}$ must be uncountable.
There is a maximal antichain of size $2^\mathfrak{c}$, but nothing more is known.
Every well-ordered ascending chain in $\mathcal{W}$ of countable cofinality has an upper bound.
For every $\kappa \leq \mathfrak{c}$ of uncountable cofinality, there is an ascending chain in $\mathcal{W}$ of type κ without upper bound.
(This is open for $\mathfrak{c} < \kappa \leq 2^\mathfrak{c}$.)

In 2012, Brattka, Gherardi and Marcone introduced an operation they called jump on partial multi-valued functions.

Their *jump* f' of f takes as input a sequence of elements of $\mathbb{N}^{\mathbb{N}}$ converging to some input for f; and the output is the output of f.

For many f that they were interested then, they proved $f <_{\mathrm{W}} f'$, but especially at higher levels of the Weihrauch lattice, there are quite a few examples of interesting g such that $g' \equiv_{\mathrm{W}} g$:
So the jump sometimes does not "jump" (i.e., strictly increase)!

Moreover, the jump is not degree-theoretic:
There exist f and g such that $f \equiv_{\mathrm{W}} g$ while $f' \not\equiv_{\mathrm{W}} g'$.

So the jump is not *weakly monotone*: There exist f and g such that $f \leq_{\mathrm{W}} g$ while $f' \not\leq_{\mathrm{W}} g'$.

(However, the jump is weakly monotone with respect to *strong* Weihrauch reducibility.)

So Marcone and Valenti started looking for a "better-behaved" jump operation.

For a partial order $\mathcal{P} = (P, \leq_P)$, we define a *jump operation* on $\mathcal{P}$ to be a function $j : P \to P$ that is

1. *strictly increasing*, i.e., $p <_P j(p)$ for every $p \in P$; and
2. *weakly monotone*, i.e., if $p \leq_P q$ then $j(p) \leq_P j(q)$ for all $p, q \in P$.

Using the Axiom of Choice, one can show that every upper semilattice without maximum, and in particular the Weihrauch lattice, has an "abstract" jump operation.

But we wanted a "natural" jump operation on the Weihrauch lattice that satisfies the above conditions.

Note: In preliminary work with Raghavan, I have a sketch of an argument that ZF alone does not suffice to ensure a jump operation on every upper semilattice without maximum.

We first need the following

Definition (totalization)

The *totalization* (or *total continuation*) of $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ is the total (multi-valued) function $\mathsf{T}f : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ defined by

$$\mathsf{T}f(p) = \begin{cases} f(p) & \text{if } p \in \mathrm{dom}(f); \\ \mathbb{N}^\mathbb{N} & \text{otherwise.} \end{cases}$$

The totalization has been before used; e.g., the Weihrauch degree of $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ is relevant for functions related to the system ATR_0.

The totalization is not a degree-theoretic operation and, even for non-total f, it is possible that $\mathsf{T}f \equiv_\mathrm{W} f$.

The totalization comes close to giving a natural jump operation, but it requires a small tweak:

Definition (totalizing jump)

The Weihrauch degree of the *totalizing jump*, or *tot-jump*, of f is the maximum of the degrees of $\mathsf{T}g$ for all $g \equiv_{\mathrm{W}} f$.
(We denote it by $\mathrm{tJ}(f)$.)

This maximum exists since it can also be defined as follows:
For $x = (e, i)^\frown p \in \mathbb{N}^{\mathbb{N}}$, define

$$\mathrm{tJ}(f)(x) = \begin{cases} \{\, \Phi_i(p, q) \mid q \in f\Phi_e(p) \,\} & \text{if } \Phi_e(p) \in \mathrm{dom}(f) \text{ and} \\ & \quad (\forall q \in f\Phi_e(p))(\Phi_i(p, q)\downarrow) \\ \mathbb{N}^{\mathbb{N}} & \text{otherwise} \end{cases}$$

This is indeed a jump operation:

Theorem

1. *For every f, $f <_{\mathrm{W}} \mathrm{tJ}(f)$.*
2. *For every f and g, if $f \leq_{\mathrm{W}} g$ then $\mathrm{tJ}(f) \leq_{\mathrm{W}} \mathrm{tJ}(g)$.*

Proof Sketch: $f \leq_{\mathrm{W}} \mathrm{tJ}(f)$ and weak monotonicity are easy.
To show $\mathrm{tJ}(f) \not\leq_{\mathrm{W}} f$, define first $d : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$d(p)(0) := p(0) + 1, \text{ and } d(p)(n) := p(n) \text{ for } n > 0.$$

Then define a new function $f_d \equiv_{\mathrm{W}} \mathrm{tJ}(f)$ (this requires a proof!) by

$$f_d(x) := \begin{cases} \{\, d\Phi_i(x,q) \mid q \in f\Phi_e(x)\,\} & \text{if } \Phi_e(x) \in \mathrm{dom}(f) \text{ and} \\ & \quad (\forall q \in f\Phi_e(x))(\Phi_i(x,q)\downarrow), \\ \mathbb{N}^{\mathbb{N}} & \text{otherwise.} \end{cases}$$

Suppose $f_d \leq_{\mathrm{W}} f$ via Φ_e and Φ_i. For $p \in \mathbb{N}^{\mathbb{N}}$, let $y = (e,i)^\frown p$.
So $y \in \mathrm{dom}(f_d)$; $\Phi_e(y) \in \mathrm{dom}(f)$; and $(\forall q \in f\Phi_e(y))(\Phi_i(y,q)\downarrow)$.

Now set $X = \{\, \Phi_i(y,t) \mid t \in f\Phi_e(y)\,\} \neq \emptyset$, so there is $q \in f\Phi_e(y)$ with $\Phi_i(y,q) \notin d(X) = \{\, d\Phi_i(y,t) \mid t \in f\Phi_e(y)\,\} = f_d(y)$. □

Another characterization of the totalizing jump is as follows:

Definition

Let $W_{\Pi^0_2\to\Pi^0_1} :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ be the problem defined as

$$W_{\Pi^0_2\to\Pi^0_1}(p) := \{q \in \mathbb{N}^\mathbb{N} \mid \forall i\,(q(i+1) > q(i) \text{ and } p(q(i)) = 0)\}.$$

So the domain of $W_{\Pi^0_2\to\Pi^0_1}$ is the set of all functions p with infinitely many zeroes, and the outputs are increasing listings of positions of infinitely many of these zeroes.
(So $W_{\Pi^0_2\to\Pi^0_1}$ converts a Π^0_2-question into a Π^0_1-question.)

Theorem

For every f, we have $\mathrm{tJ}(f) \equiv_{\mathrm{W}} \mathsf{T}(W_{\Pi^0_2\to\Pi^0_1} * f * W_{\Pi^0_2\to\Pi^0_1})$.
(Here, $g * h$ is the sequential product of h and then g.)

But $W_{\Pi^0_2\to\Pi^0_1}$ cannot be dropped on either side of f!

We have far more than for "usual jumps" on degree structures:

Theorem

For every f, g, $f \leq_{\mathrm{W}} g$ if and only if $\mathrm{tJ}(f) \leq_{\mathrm{W}} \mathrm{tJ}(g)$.
Thus tJ is an (injective!) endomorphism on the Weihrauch degrees.

Here are some specific examples of totalizing jumps:

Theorem

- $\mathrm{tJ}(\emptyset) = \mathrm{id}$;
- $\mathrm{tJ}(\mathrm{id})(p) = \{\, q \mid \exists m \forall n > m\, p(n) = 1 \iff \exists n\, q(n) > 0 \,\}$;
- $\mathrm{tJ}(\mathrm{C}_{\mathbb{N}^\mathbb{N}}) = \mathrm{TC}_{\mathbb{N}^\mathbb{N}}$.

Moreover, $\mathrm{LPO} <_{\mathrm{W}} \mathrm{tJ}(\mathrm{id})$ *and* $\widehat{\mathrm{tJ}(\mathrm{id})} \equiv_{\mathrm{W}} \lim$.
(Here $\mathrm{LPO} : \mathbb{N}^\mathbb{N} \to \{0, 1\}$ *is defined by* $\mathrm{LPO}(p) = 0$ *iff* $p \neq 0^\mathbb{N}$;
and $\hat{f}$ *is the countable parallelization of* f*.)*

We also have explicit descriptions of $\mathrm{tJ}^n(\mathrm{id})$ for every n.

Main Open Question

Characterize the range of the totalizing jump on the Weihrauch degrees.

We are very far from a satisfactory answer.
In fact, while our definition of the totalizing jump is Δ^1_2,
we would like to know if we can lower that complexity to Π^1_1, say,
given that we can prove it cannot be Σ^1_1.

But we do have some interesting properties of the totalizing jump:

Theorem

For every f, $\mathrm{tJ}(f)$ is total and join-irreducible but can be meet-reducible.

However, not all total join-irreducible Weihrauch degrees are in the range of tJ, e.g., (the degree of) $\mathrm{TC}_{\mathbb{N}}$ is not in the range of tJ.

Here are some more properties of the totalizing jump:

Theorem

For every $g \neq \emptyset$*, there exists* $f <_W g$ *such that* $\mathrm{tJ}(f) \not\leq_W g$.

Theorem

For every f*, there exists* h *s.t.* $f <_W h <_W \mathrm{tJ}(f)$.

Theorem

If $\mathrm{LPO} \times g \leq_W \mathrm{tJ}(f)$*, then* $g \leq_W f$.
In fact (with Pauly): If $\mathrm{DIS} \times g \leq_W \mathrm{tJ}(f)$*, then* $g \leq_W f$.
(Here, $\mathrm{DIS} : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}, p \mapsto \{ q \mid U(p) \neq q \}$ *for a fixed universal partial computable functional* $U :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$*.)*

Corollary

If $\mathrm{DIS} \times g \leq_W g$, then g is not in the range of tJ.

Thank you!