Some results on enumeration Weihrauch reduction

Alice Vidrine (joint with Mariya Soskova)

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Introduction: What

Problems and Weihrauch reduction

Weihrauch reduction is a way of comparing the computational strength of various "problems", represented as partial multifunctions on $\mathbb{N}^{\mathbb{N}}$.

We may think of Weihrauch reduction $f \leq_W g$ as a computation of values of f, given the ability to query g as an oracle *exactly once*.

Formally, we have this reduction if there are computable functionals (Φ, Ψ) such that

1. $\alpha \in \operatorname{dom} f \Rightarrow \Phi(\alpha) \in \operatorname{dom} g$
2. for any $\alpha \in \operatorname{dom} f$ and $\beta \in g(\Phi(\alpha))$, we have $\Psi(\alpha, \beta) \in f(\alpha)$.

The algebras $\mathbb{P}$ & $\mathbb{P}_\sharp$

Define $\mathbb{P}$ to be $\mathcal{P}(\mathbb{N})$ equipped with a binary operation (called *application*) given by

$$AB = \{n : \exists m(\langle n, m\rangle \in A \wedge D_m \subseteq B)\}.$$

We read application as left associative (e.g. ABC means $(AB)C$.)

The algebra $\mathbb{P}_\sharp$ is the substructure of $\mathbb{P}$ consisting of the c.e. sets (i.e. enumeration operators).

Dana Scott proved in [Scott, 1976] that both these algebras can interpret the untyped lambda calculus or Schönfinkel/Curry's combinator calculus.

eW-problems and eW-reductions

An **e**W-problem is a partial multifunction from $\mathbb{P}$ to itself. Given problems f, g, we say that $f \leqslant_{\mathbf{e}W} g$ if there are enumeration operators Γ, Δ such that

1. if $A \in \operatorname{dom} f$ then $\Gamma A \in \operatorname{dom} g$,
2. and for any $A \in \operatorname{dom} f$ and $X \in g(\Gamma A)$, $\Delta\langle A, X\rangle \in f(A)$.

In other words, **e**W-reduction is just Weihrauch reduction where the problems operate on $\mathbb{P}$, and enumeration reduction (i.e. the action of elements in $\mathbb{P}_\sharp$) is our notion of computation.

Introduction: Why

Why formulate the notion of eW-reduction?

First, enumeration operators have a robust computational structure, and their use to study problems-as-multifunctions is intrinsically interesting.

Moreover, it's a notion of computation that works on *positive* information, potentially making some different distinctions between common problems.

Why formulate the notion of eW-reduction?

Second, they are related to an under-studied *realizability topos*.

- A topos is a category theoretic model of a kind of intuitionistic set theory. Realizability toposes are built from a model of computation (see [van Oosten, 2008] for an overview of the area).
- There is a realizability topos where the underlying model of computation is enumeration reduction—the topos $\mathrm{RT}(\mathbb{P}, \mathbb{P}_\sharp)$.
- There is a strong relationship between $\mathcal{D}_{\mathbf{e}W}$ and subtoposes of $\mathrm{RT}(\mathbb{P}, \mathbb{P}_\sharp)$ (see [Kihara, 2023]).

Basic results about eW-reduction

The eW degrees extend the Weihrauch

Proposition. There is an embedding of the Weihrauch degrees into the **e**W degrees.

Proof sketch.

- Using the injective function $\mathrm{gr} : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$, replace a Weihrauch problem f with $\tilde{f}$ so that $\mathrm{gr}(\alpha) \in \tilde{f}(\mathrm{gr}(\beta))$ iff $\alpha \in f(\beta)$.
- We can replace each Turing functional in a reduction $f \leq_W g$ with enumeration operators that witness $\tilde{f} \leqslant_{\mathbf{e}W} \tilde{g}$. (Think about the graph of the computable function $\omega^{<\omega} \to \omega^{<\omega}$ that defines the functional.)
- Now we want $\tilde{f} \leqslant_{\mathbf{e}W} \tilde{g}$ to imply $f \leq_W g$. Given an enumeration operator Γ, we may pick a computable enumeration γ of Γ and define a functional Φ such that $\Phi(\alpha)(n)$ is found by searching longer and longer portions of Γ and α to find when $\Gamma(\mathrm{gr}(\alpha))$ outputs a pair $\langle n, k\rangle$.

The eW degrees extend the Weihrauch

This mapping is not surjective.

Let $g :\subseteq \mathbb{P} \rightrightarrows \mathbb{P}$ have domain consisting of a single 1-generic G, to which every element of $\mathbb{P}$ is a solution. Suppose that there is a Weihrauch problem f with $g \equiv_{\mathbf{e}W} \tilde{f}$.

Since G is quasi-minimal, and every element in the domain of $\tilde{f}$ is total, $\Gamma : \mathrm{dom}\, g \to \mathrm{dom}\, \tilde{f}$ occurring in a reduction $g \leqslant_{\mathbf{e}W} \tilde{f}$ must send G to a computable element.

But now a reduction $\tilde{f} \leqslant_{\mathbf{e}W} g$ must send that computable element of $\mathrm{dom}\, \tilde{f}$ to G, requiring G to be a c.e. set, contradicting 1-genericity.

The problem *id*

Definition. The problem *id* is the identity function on $\mathbb{P}$.

Proposition.

1. $f \leqslant_{\mathbf{e}W} id$ if and only if there is an enumeration operator Γ such that for all $A \in \operatorname{dom} f$, $\Gamma A \in f(A)$.
2. $id \leqslant_{\mathbf{e}W} f$ if and only if f has a c.e. instance.

Proof.

1. Let the reduction be witnessed by (Γ, Δ); then $\Delta(A, \Gamma A) \in f(A)$ and can be coded by a single enumeration operator.
2. $\varnothing$ is an *id*-instance, so if (Γ, Δ) witness a reduction, $\Gamma\varnothing$ must be a (c.e.) f-instance.

Closed choice problems

Definition. A computable metric space X is a separable metric space with a listing of a dense set $(p_n)_{n\in\omega}$ such that the distance function $(n, m) \to d(p_n, p_m)$ is computable.

The *closed choice problem* on a complete metric space (or some non-metric spaces) X, $\mathsf{C}_X :\subseteq \mathbb{P} \rightrightarrows \mathbb{P}$, takes an encoding of an open set with non-empty complement, and returns (the name of) an element of that complement.

Closed choice problems

In the Weihrauch setting, open complements of closed sets are coded by enumerations of open balls—i.e. of pairs (n, r) representing an open ball of radius r around p_n.

In the $\mathbf{e}W$ setting, we may simply take the *set* of open balls instead of a listing of the open balls. In the case of Baire or Cantor space, we may equivalently represent open sets by sets of finite strings, and for $\mathbb{N}$ we represent open sets by themselves.

A first example: $\mathsf{C}_{\mathbb{P}}$

A natural topology on $\mathbb{P}$ is the *positive information topology*: the basic open sets are of the form $O_a := \{A \in \mathcal{P}(\mathbb{N}) : a \subseteq A\}$ for a finite. (Note: this isn't actually a metric space.)

We may represent open sets O by a set $I \subseteq \mathbb{N}$ such that $O = \bigcup_{i \in I} O_{D_i}$.

Proposition. $\mathsf{C}_{\mathbb{P}} \equiv_{\mathbf{e}W} id$.

Proof. Every closed set of $\mathbb{P}$ contains $\varnothing$, so the enumeration operator coding $\lambda x.\varnothing$ computes solutions from instances.

The $\mathbf{e}W$ versions of choice problems tend to fall strictly above their Weihrauch counterparts.

For instance, $\widetilde{\mathsf{C}_{\mathbb{N}}} <_{\mathbf{e}W} \mathsf{C}_{\mathbb{N}}$, the reduction being easy—one need only take the graph of a function to its range, and take the $\mathsf{C}_{\mathbb{N}}$-solution $\{m\}$ to the graph of the constant function with value m.

On the other hand, $\varnothing$ is an instance of $\mathsf{C}_{\mathbb{N}}$, so consider two distinct singleton closed sets A, B. Since $\varnothing \subseteq A, B$, we must have $\Gamma\varnothing \subseteq \Gamma A, \Gamma B$ for any enumeration operator Γ; but the image of Γ consists of graphs of total functions, so it has to be constant.

The arguments for $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ and $\mathsf{C}_{2^{\mathbb{N}}}$ are similar.

$\mathsf{C}_{\mathbb{N}}$ & $\mathsf{UC}_{\mathbb{N}}$

The first interesting separation that develops in the $\mathbf{e}W$ setting concerns $\mathsf{C}_{\mathbb{N}}$ and its restriction to singletons, $\mathsf{UC}_{\mathbb{N}}$.

Fact. $\widetilde{\mathsf{C}_{\mathbb{N}}} \equiv_{\mathbf{e}W} \widetilde{\mathsf{UC}_{\mathbb{N}}}$.

Proposition. $\mathsf{UC}_{\mathbb{N}} <_{\mathbf{e}W} \mathsf{C}_{\mathbb{N}}$

$\mathsf{C}_{\mathbb{N}}$ & $\mathsf{UC}_{\mathbb{N}}$

Proof. The reduction is immediate from the fact that unique choice is just a restriction of closed choice.

For strictness, suppose we had a reduction $\mathsf{C}_{\mathbb{N}} \leqslant_{\mathbf{e}W} \mathsf{UC}_{\mathbb{N}}$ witnessed by (Γ, Δ). Then $\Gamma\varnothing$ is the complement of a singleton, and for any other $A \in \operatorname{dom} \mathsf{C}_{\mathbb{N}}$, ΓA must both be the complement of a singleton, and a superset of $\Gamma\varnothing$. The only way this can happen is if $\Gamma A = \Gamma\varnothing$, meaning Γ must be constant.

Now consider $\{k\} = \Delta(\varnothing \oplus \{n\})$, where $n \in \overline{\Gamma\varnothing}$; then $\Delta(\{k\} \oplus \{n\})$ must also contain k by monotonicity, and cannot be outputting any subset of $\overline{\{k\}}$.

Commentary

In general, we think of instances and solutions as codes for elements of mathematical objects (e.g. points in spaces, or closed sets of topologies).

Here we see a substantial difference in behavior depending on *what* information our codes contain—positive and negative information, or just positive.

I don't know what this means, but it's pretty cool.

$C_{2^{\mathbb{N}}}$ **and** WKL

We have $\widetilde{C_{2^{\mathbb{N}}}} \equiv_{\mathbf{e}W} \widetilde{\mathsf{WKL}}$ as a standard result from the Weihrauch degrees. With positive and negative information, these are two different representations of the same thing.

On the other hand,

Proposition. $C_{2^{\mathbb{N}}} \mid_{\mathbf{e}W} \mathsf{WKL}$

$\mathsf{C}_{2^{\mathbb{N}}}$ **and** WKL

Proof. Suppose in each case below that (Γ, Δ), towards a contradiction, witnesses the specified reduction.

1. ($\mathsf{C}_{2^{\mathbb{N}}} \not\leq_{\mathbf{e}W}$ WKL). Consider the set $\Gamma\varnothing$ ($\varnothing$ coding the full closed set of Cantor space). Then $\Gamma\varnothing$ is an infinite c.e. tree, such that for any $\mathsf{C}_{2^{\mathbb{N}}}$-instance C we have $\Gamma\varnothing \subset \Gamma C$. So there is a $\mathbf{0}''$-computable path P such that $\Delta(C \oplus P) \in \mathsf{C}_{2^{\mathbb{N}}}(C)$ for any $C \in \mathrm{dom}\mathsf{C}_{2^{\mathbb{N}}}$.

   Now let C be the complement of a $\mathbf{0}''$-computable tree with no $\mathbf{0}''$-computable paths. Then $\Delta(C \oplus P) \leq_e C \oplus P \leq_T \mathbf{0}''$. Since $\Delta(C \oplus P)$ is a total object (elements of Cantor space are total functions), this makes it $\mathbf{0}''$-computable, which is impossible if (Γ, Δ) is a reduction.

$\mathsf{C}_{2^{\mathbb{N}}}$ **and** WKL

Proof. Suppose in each case below that (Γ, Δ), towards a contradiction, witnesses the specified reduction.

2. (WKL $\not\leq_{\mathbf{e}W}$ $\mathsf{C}_{2^{\mathbb{N}}}$). Consider the full tree $T = 2^{<\omega}$, and the closed set ΓT. Again, ΓT, as a set of strings coding the complement of a closed set, is c.e., so the closed set in question is Π^0_1; moreover, it's a subset of every other closed set in the image of Γ.

   So we fix a Δ^0_2 element $P \in \Gamma T$. Now choose a Δ^0_2 tree T with no Δ^0_2 paths, and proceed as above.

In fact, for very simlar reasons, we even have WKL $|_{\mathbf{e}W}$ $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$!

Commentary

We've now seen that we can see that an equivalence from the Weihrauch setting may break down in both directions, or only one.

The trend in these proofs of exploiting $\subseteq$-monotonicity of enumeration operators is common to many proofs in the setting.

Some generalities about *WKL* & closed choice

WKL **can't reduce to any closed choice problem.** We can generalize the previous instances. This plays on the fact that the set of all trees has a largest element under inclusion, and we can make trees of any complexity with even more complex paths.

Unique choice problems are always weaker. There's nothing special about $\mathbb{N}$ in our analysis of $C_{\mathbb{N}}$ and $UC_{\mathbb{N}}$.

The Weihrauch degrees and eW degrees

The W / eW relationship

Are the Weihrauch and eW degrees non-isomorphic? Is there a first order difference between them?

The answer to this turns out to be affirmative. We start with the fact that the Weihrauch degree of *id* is definable as the greatest strong minimal cover [Lempp et al., 2023], and the degrees below *id* are isomorphic to the dual of the Medvedev degrees.

If *id* is definable in the **e**W degrees with the same definition, then we may move the question to a simpler set of problems.

The W / $\mathbf{e}W$ relationship

Definition(s). Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{P}$, and define $\mathcal{X} \leq_D \mathcal{Y}$ if and only if there is an enumeration operator Γ such that for all $A \in \mathcal{Y}$, $\Gamma A \in \mathcal{X}$. We call the degree system induced by $\leq_D$ the *Dyment degrees*, $\mathcal{D}_D$.

Note. This is not the definition of the Dyment degrees typically found in the literature.

It's not too hard to see that the $\mathbf{e}W$ degrees below *id* are isomorphic to the dual of $\mathcal{D}_D$.

The W / eW relationship

Theorem. The degree of *id* is definable in the eW degrees.

The proof will go through the following lemmas:

1. If $\mathbf{a} \in \mathcal{D}_{\mathbf{e}W}$ is a strong minimal cover, it contains a problem with finite domain.
    1.1 There is a problem, equivalent to the one we will construct above, with singleton domain.
2. No problem f with singleton domain and $f \nleq_{\mathbf{e}W} id$ is a strong minimal cover.

The W / eW relationship

Lemma 1

If $\mathbf{a} \in \mathcal{D}_{\mathbf{eW}}$ is a strong minimal cover, it contains a problem with finite domain.

Proof

The proof proceeds similarly to that in [Lempp et al., 2023]. Suppose that $\deg(f)$ is a strong minimal cover of $\deg(h)$. We will construct a problem $g = \bigcup_{s\in\omega} g_s$ by stages such that $\operatorname{dom} g \subseteq \{\langle\{n\}, A\rangle \;:\; n \in \mathbb{N} \wedge A \in \operatorname{dom} f\}$, and if g had infinite domain, we would have $g <_{\mathbf{e}W} f$ and $g \nleq_{\mathbf{e}W} h$. When the construction fails, we have the desired finite domain problem.

For simplicity, we will treat partial multifunctions as binary relations on $\mathbb{P}$.

The W / eW relationship

Proof (cont.)

Stage 0: $g_0 = \varnothing$

Stage $s+1 = 2\langle e, i\rangle$: Here we ensure that (Γ_e, Γ_i) is not a reduction $g \leqslant_{\mathbf{e}W} h$. Let $k \in \mathbb{N}$ be a number which is not banned (see below) and is greater than any other occurring as the first component of a pair in $\mathrm{dom}\, g_s$, and let $\hat{g} = g_s \cup \{\langle\{k\}, A, B\rangle \;:\; B \in f(A)\}$.

Since $\hat{g}$ is equivalent to f, we either have some $\langle\{n\}, A\rangle \in \mathrm{dom}\, \hat{g}$ with $\Gamma_e\langle\{n\}, A\rangle \notin \mathrm{dom}\, h$ or there is a $B \in h(\Gamma_e\langle\{n\}, A\rangle)$ such that $\Gamma_i(A, B) \notin \hat{g}(\langle\{n\}, A\rangle)$. If we have such a $\langle\{n\}, A\rangle$, we let $g_{s+1} = g_s \cup \{\langle\{n\}, A, X\rangle \;:\; X \in f(A)\}$.

Proof (cont.)

Stage $s+1 = 2\langle e, i \rangle + 1$: Here we try to ensure $f \not\leqslant_{\mathbf{e}W} g$. If (Γ_e, Γ_i) are a reduction $f \leqslant_{\mathbf{e}W} g_s$, we have our $g = g_s \in \deg(f)$. Otherwise, there is some f-instance A on which (Γ_e, Γ_i) fails as a reduction. If $\Gamma_e A \neq \langle \{n\}, B \rangle$ for some n, or if it's in $\mathrm{dom}\, g_s$, we do nothing. If $\Gamma_e A = \langle \{n\}, B \rangle \notin \mathrm{dom}\, g_s$, then we ban n from being used in any future even step. □

Note that the odd steps are the only steps on which our diagonalizing efforts can fail.

The W / eW relationship

Lemma 1.1

The g constructed in the previous lemma is equivalent to a problem with singleton domain.

Proof

If $\operatorname{dom} g = \{\langle\{n_0\}, A_0\rangle, \ldots, \langle\{n_k\}, A_k\rangle\}$, let $g' = g \restriction \{\langle\{n_0\}, A_0\rangle\}$ and let $g'' = g \restriction \{\langle\{n_1\}, A_1\rangle, \ldots, \langle\{n_k\}, A_k\rangle\}$. Since g is constructed in such a way that its domain contains at most one pair $\langle\{n\}, A\rangle$ for each n, we can build an enumeration operator that witnesses $g' \sqcup g'' \equiv_{\mathsf{e}W} g$.

Since deg(g) is join-irreducible, one of g' or g'' is equivalent to g. If it's g', we are done. If it's g'', we decompose it similarly; by induction, we will eventually find the desired singleton-domain problem. □

Lemma 2

The degree of the problem *id* is the greatest strong minimal cover.

Proof

Suppose we have problems f, h where f has singleton domain$\{$A$\}$, $f \nleqslant_{\mathbf{e}W} h$, and $f \nleqslant_{\mathbf{e}W} id$. Then we can build a set D such that $f \sqcap \chi_D <_{\mathbf{e}W} f$ and $f \sqcap \chi_D \nleqslant_{\mathbf{e}W} h$.

Let $D_0 = \varnothing$. At stage $s + 1 = \langle e, i \rangle$, we see if $f \sqcap \chi_{D_s} \leqslant_{\mathbf{e}W} h$. If not, we needn't do anything. If so, note that if

$$\Gamma_i(\langle A, \{n\} \rangle, X) = \langle \{0\}, B \rangle$$

with $B \in f(A)$, for any n, then we can construct a reduction $f \leqslant_{\mathbf{e}W} h$.

The W / eW relationship

Proof (cont.)
So we must have

$$\Gamma_i(\langle A, \{n\}\rangle, X) = \langle \{1\}, j\rangle$$

for all $f \sqcap \chi_{D_s}$-instances and $X \in h(\Gamma_e\langle A, \{n\}\rangle)$, with $j \in \{0, 1\}$. Then we add s to D_{s+1} (or don't) so as to ensure $\chi_{D_{s+1}}$ disagrees on the s-th bit with χ_{D_s}. We continue this way, and set $D = \bigcup_{s\in\omega} D_s$.

Note that we cannot have $f \leqslant_{\mathbf{e}W} f \sqcap \chi_D$. If we did have such a reduction (Γ, Δ), then $\Delta(A, \langle\{1\}, \{j\}\rangle) \in f(A)$ for $\langle\{1\}, \{j\}\rangle \in f \sqcap \chi_D(\Gamma A)$. But then $A \mapsto \Delta(A, \langle\{1\}, \{j\}\rangle)$ is an enumeration reduction taking our f-instance to a solution, which would imply $f \leqslant_{\mathbf{e}W} id$. □

Conclusion

In fact, the degrees of $\mathbf{e}W$-problems below *id* with singleton domains are exactly the strong minimal covers, which also constitute the dual of the enumeration degrees.

Now we can show a first order difference between the Weihrauch and $\mathbf{e}W$ degrees. By the above and Theorem 2.1 in [Dyment, 1976], the strong minimal covers in each are isomorphic to the duals of the Turing and enumeration degrees, respectively, which differ on the dualization of the statement "there exists a minimal degree."

Thanks!

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