Erdős-Hajnal property in NIP theories

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Definitions

Definition

A *graph* is a pair $G = (V, E)$ where V is a finite set, E is a binary symmetric anti-reflexive relation on V. An element $x \in V$ is called a *vertex*. A pair $\{x, y\} \in E$ is called an *edge*.

We say x and y are *adjacent* if $\{x, y\} \in E$.

A *clique* in G is a set of vertices all pairwise adjacent.

An *anticlique* in G is a set of vertices all pairwise non-adjacent.

A *homogeneous set* means a clique or an anticlique.

Erdős-Hajnal Conjecture

Definition

Given a graph H, we say that a graph G is H-free if G has no induced subgraph isomorphic to H.

Erdős-Hajnal Conjecture [EH89]:

Conjecture

For any graph H there is $\epsilon > 0$ such that if a graph G does not contain any induced subgraph isomorphic to H then G has a clique (i.e. every two distinct vertices are adjacent) or an anti-clique (i.e. every two distinct vertices are nonadjacent) of size $\geq |G|^{\epsilon}$.

Background

The conjecture originates from Ramsey's theorem, which says for any fixed m, any graph large enough has a homogeneous subset (i.e. a clique or an anti-clique) of size m in the graph:

Fact

[Ram87, Theorem B] For all $m \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that for any graph G with size $\geq n$, then G has a homogeneous set of size m.

Background

It was known that given a graph, a homogeneous subset of "log" size always exists:

Fact

[ES35] All graphs G with $|G| = n$ contain a homogeneous set of size at least $\dfrac{\log n}{2 \log 2}$.

Background

Can one find a homogeneous subset as large as of "polynomial" size in general? The answer is no.

Fact

[Erd47, Theorem I] For all sufficiently large n there is a graph G not containing homogeneous sets of size $\frac{2\log n}{\log 2}$.

Background

But if a graph omits some small graph, then homogeneous subsets of "polynomial" size exist. For example, if H is a cograph, then Erdős-Hajnal conjecture holds for H-free graphs. We define the family $\mathcal{C}$ of *cographs*:

1. The graph with a single vertex is a cograph;
2. If $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are cographs, then $G = (V_1 \dot{\cup} V_2, E_1 \dot{\cup} E_2)$ is a cograph;
3. If $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are cographs, then $G = (V_1 \dot{\cup} V_2, E_1 \dot{\cup} E_2 \dot{\cup} \{\{v, w\} : v \in V_1, w \in V_2\})$ is a cograph.

Background

Fact

[EH89, Theorem 1.2] If H is a cograph, then there is $\epsilon = \epsilon(H) > 0$ such that if G is H-free, then G has a homogeneous set of size $\geq |G|^{\epsilon}$.

Since graphs omitting some fixed small graph behave nicely, it is natural to conjecture that for any graph H, if a graph G omits H, then G has a homogeneous subset of size at least $|G|^{\epsilon}$ where ϵ depends on H only.

Background

A family of graphs is called *hereditary* if for any graph G in the family, all of its induced subgraphs are in the family. It's obvious that for any graph H, the family of H-free graphs is hereditary.

In general, we say a family of graphs has *Erdős-Hajnal property* if there is $\epsilon > 0$ such that for any graph G in the family, there is a homogeneous subset in G of size $\geq |G|^\epsilon$.

Question

Which family of graphs has Erdős-Hajnal property?

Plan

In this talk, we mainly talk about Erdős-Hajnal property in NIP theory.

Plan:

1. We define NIP.
2. We look at Erdős-Hajnal property in stable theories and distal theories, which are subcases of NIP theories.
3. My results on Erdős-Hajnal property in special NIP cases:
   - how to prove Erdős-Hajnal property for the family of graphs with VC-dimension ≤ 2 using model theory and combinatorics;
   - we show that strong Erdős-Hajnal property holds for the family of graphs with bounded VC-minimal complexity;
   - a lemma I found in my attempt to prove Erdős-Hajnal property for the family of dp-minimal graphs.
4. Recently, Erdős-Hajnal property of NIP graphs was proved by Nguyen, Scott, and Seymour in [NSS23]. We will briefly introduce their work.

NIP

Definition

[Sim15] Let T be a complete theory. Let $\mathcal{U}$ be a monster model of T. Let $\varphi(\bar{x};\bar{y})$ be a partitioned formula. We say that a set A of $|\bar{x}|$-tuples is *shattered* by $\varphi(\bar{x};\bar{y})$ if we can find a family $(b_I : I \subseteq A)$ of $|\bar{y}|$-tuples such that for all $a \in A$

$$\mathcal{U} \models \varphi(a; b_I) \Leftrightarrow a \in I.$$

A partitioned formula $\varphi(\bar{x};\bar{y})$ is *NIP* (or dependent) if no infinite set of $|\bar{x}|$-tuples is shattered by $\varphi(\bar{x};\bar{y})$.

The maximal integer n for which there is some A of size n shattered by $\varphi(\bar{x};\bar{y})$ is called the *VC-dimension* of φ.

[Sim15, Definition 2.10.] The theory T is *NIP* if all formulas $\varphi(\bar{x};\bar{y}) \in \mathcal{L}$ are NIP.

[Sim15, Remark 2.3.] If $\varphi(\bar{x};\bar{y})$ is NIP, then by compactness, there is some integer n such that no set of size n is shattered by $\varphi(\bar{x};\bar{y})$.

Stable case

Malliaris and Shelah proved in [MS14] that for every $k \in \mathbb{N}$ the family of k-stable graphs have Erdős-Hajnal property.

Definition

Let $G = (V, E)$ be a graph. Let $k \in \mathbb{N}$. G is *k-stable* if there do not exist some $a_1, ..., a_k \in V$, $b_1, ..., b_k \in V$ such that $E(a_i, b_j)$ holds if and only if $i \leq j$.

It is easy to see from the definition of k-stability that

Fact

For any $k \in \mathbb{N}$, a k-stable graph has bounded VC-dimension.

It was shown in [MS14] that Erdős-Hajnal property holds for k-stable graphs.

Fact

[MS14, Theorem 3.5 (2)] For $k \in \mathbb{N}$, there is $\epsilon_k > 0$ such that for any k-stable graph G, G has a homogeneous subset of size $\geq |G|^{\epsilon_k}$

Stable case-another proof

Chernikov and Starchenko gave another proof of the same result in [CS18a] using δ-dimension technique.

Let $\{G_i = (V_i, E_i) : i \in \omega\}$ be a sequence of finite graphs. Let $\mathcal{F}$ be a non-principal ultrafilter of ω. Let $G = (V, E)$ be the ultraproduct $\prod_{i \in \omega} (V_i, E_i)/\mathcal{F}$.

Definition

Let A be an internal set $\prod_{i \in \omega} A_i/\mathcal{F}$, where each A_i is a non-empty subset of V_i. For each $i \in \omega$, let $l_i = \log(|A_i|)/\log(|V_i|)$. We define the *$\delta$-dimension of A*, denoted by $\delta(A)$, to be the unique number $l \in [0, 1]$ such that for any $\epsilon \in \mathbb{R}^{>0}$, the set $\{i \in \omega : l - \epsilon < l_i < l + \epsilon\}$ is in $\mathcal{F}$.

Then that Erdős-Hajnal property holds for the family $\{G_i : i < \omega\}$ is the same as there is a homogeneous set $A \subseteq G := \prod_{i \in \omega} G_i/\mathcal{F}$ with $\delta(A) > 0$.

Stable case-another proof

In [CS18a], using Shelah's 2-rank, Chernikov and Starchenko showed that for each $k \in \mathbb{N}$, if $\{G_i : i < \omega\}$ is a family of k-stable graphs and $\mathcal{F}$ is a non-principal ultrafilter on ω, then there is an internal homogeneous set $A \subseteq G$ such that $\delta(A) > 0$.

Results using δ-dimension

(F.)

1. A slightly different proof using the same δ-dimension technique that for each $k \in \mathbb{N}$ Erdős-Hajnal property holds for the family of k-stable graphs.
2. We also show that δ-dimension technique combined with substitution proves that the family of graphs with VC-dimension ≤ 2 has Erdős-Hajnal property.
3. Also as an application of δ-dimension, we can prove without using the substitution technique that Erdős-Hajnal property holds for the family of graphs with VC-dimension 1.

Using δ-dimension

Definition

Let $G = (V, E)$ be the ultraproduct $\prod_{i \in \omega} V_i / \mathcal{F}$. For a definable set $A \subseteq V$ such that $\delta(A) > 0$, we say that A satisfies *Property* $(*)$ if there is a definable $A^+ \subseteq \{a \in A \mid \delta(\{x \in A \mid E(x, a)\}) < \delta(A)\}$ such that $\delta(A^+) = \delta(A)$ or there is a definable $A^- \subseteq \{a \in A \mid \delta(\{x \in A \mid \neg E(x, a)\}) < \delta(A)\}$ such that $\delta(A^-) = \delta(A)$.

Proposition

Let $G = (V, E)$ be the ultraproduct $\prod_{i \in \omega} V_i / \mathcal{F}$. Assume $A \subseteq V$ is definable with $\delta(A) > 0$, and A satisfies property $(*)$. Then A has a homogeneous subset with positive δ-dimension.

Using δ-dimension

Claim

(F.) Fix a definable A such that $\delta(A) > 0$. If property $(*)$ fails for A, i.e. if for all definable $B \subseteq A$ with $\delta(B) = \delta(A)$,

1. $B \not\subseteq \{a \in A \mid \delta(\{x \in A \mid E(x,a)\}) < \delta(A)\}$ and
2. $B \not\subseteq \{a \in A \mid \delta(\{x \in A \mid \neg E(x,a)\}) < \delta(A)\}$,

then for all $B \subseteq A$ with $\delta(B) = \delta(A)$,

$$B \not\subseteq \{a \in A \mid \delta(\{x \in A \mid E(x,a)\}) < \delta(A)\} \cup$$

$$\{a \in A \mid \delta(\{x \in A \mid \neg E(x,a)\}) < \delta(A)\}.$$

Using δ-dimension

(Claim continued.)
Moreover, suppose property $(*)$ fails for all A with $\delta(A) > 0$. Fix A with $\delta(A) > 0$. Then for any $B \subseteq A$ with $\delta(B) = \delta(A)$, there exist $a, a' \in B$, $a \neq a'$ such that

1. $\delta(\{x \in A \mid E(x, a)\}) > 0$,
2. $\delta(\{x \in A \mid \neg E(x, a)\}) > 0$,
3. $\delta(\{x \in A \mid E(x, a')\}) > 0$,
4. $\delta(\{x \in A \mid \neg E(x, a')\}) > 0$ and
5. $E(a, a')$.

Revisiting stable case

Fact

[She90, Theorem 2.2] Let $G = (V, E)$ *be the ultraproduct* $\prod_{i \in \omega} V_i / \mathcal{F}$. *G is k-unstable for all* $k \in \mathbb{N}$ *iff there is* $A \subseteq V$ *and* $\lambda \geq \aleph_0$ *such that* $|S^1_E(A)| > \lambda \geq |A|$.
$(S^1_E(A) := \{ \bigcap_{a \in A} E(x; a)^{\epsilon(\bar{a})} : \epsilon \in 2^A \}.)$

Revisiting stable case

Theorem

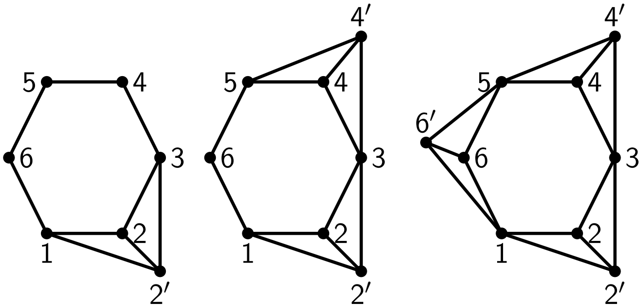
(F.) A different proof for the following fact:
For each $k \in \mathbb{N}$, the family of k-stable graphs has the Erdős-Hajnal property.

VC-dimension 2 case

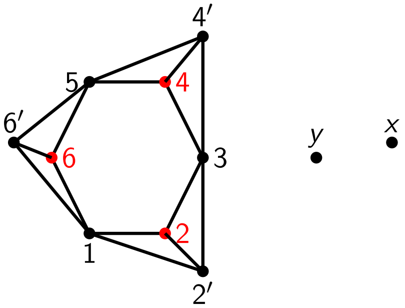
Theorem

(F.) The family of graphs with VC-dimension ≤ 2 *has the Erdős-Hajnal property.*

VC-dimension 2 case



VC-dimension 2 case



VC-dimension 1 without using substitution

Theorem

(F.)Without using substitution, one can prove the family of finite graphs with VC-dimension ≤ 1 *has the Erdős-Hajnal property.*

Distal case

Another important subcase of NIP theories is the distal theories, which are often considered as the opposite of stable theories. Chernikov and Starchenko proved in [CS18b] that the family of distal graphs has Erdős-Hajnal property by showing that *strong Erdős-Hajnal property* holds for the family.

Strong Erdős-Hajnal

Before we go into the distal case, we give the definition of Strong Erdős-Hajnal property and related facts.

Definition

A family $\mathcal{F}$ of graphs has the *strong Erdős-Hajnal Property* if there is $k > 0$ such that for every $G \in \mathcal{F}$ there exist disjoint subsets $A, B \subseteq V$ satisfying that

- $|A| \geq k|V|$ and $|B| \geq k|V|$;
- $A \times B \subseteq E$ or $A \times B \subseteq \neg E$.

Strong Erdős-Hajnal

The following fact says in order to find a homogeneous set of polynomial size, it suffices to find a cograph of polynomial size.

Fact

If G is a cograph, then G has a homogeneous set of size $\geq |G|^{\frac{1}{2}}$.

Strong Erdős-Hajnal property implies Erdős-Hajnal property by finding a cograph of polynomial size:

Fact

If a hereditary family $\mathcal{F}$ of graphs has strong Erdős-Hajnal property, then $\mathcal{F}$ has Erdős-Hajnal property.

Setting in model theory

In general, we talk about infinite models $\mathcal{M} \models T$. But Erdős-Hajnal property is about finite objects. In [CS18b], Chernikov and Starchenko gave a modified version of Erdős-Hajnal property so that we can discuss Erdős-Hajnal property in an infinite model.

Definition

[CS18b, Definition 1.5.] Let $\mathcal{M}$ be a first-order structure and $R \subseteq M^k \times M^k$ be a definable relation. Consider the family $\mathcal{G}_R$ of all finite graphs $G = (V, E)$ where $V \subseteq M^k$ is a finite subset and $E = (V \times V) \cap R$. We say that R satisfies the *(strong) Erdős-Hajnal property* if the family $\mathcal{G}_R$ does.

Distal theories

The notion of distal theories was introduced in [Sim13]. For convenience, we use the following equivalent definition.

Definition

[CS15, Theorem 21] [CS18b, Fact 2.5.] Let T be a complete NIP theory. T is *distal* if for every formula $\varphi(\bar{x}, \bar{y})$ there is a formula $\psi(\bar{x}, \bar{y}_1, ..., \bar{y}_n)$ with $|\bar{y}_1| = \ = |\bar{y}_n| = |\bar{y}|$ such that: for any $\mathcal{M} \models T$, for any finite $B \subseteq M^{|\bar{y}|}$ with $|B| \geq 2$ and any $a \in M^{|\bar{x}|}$, there are $b_1, ..., b_n \in B$ such that $\mathcal{M} \models \psi(a, b_1, ..., b_n)$ and $\psi(\bar{x}, b_1, ..., b_n) \vdash tp_{\varphi}(a/B)$ (i.e. for any $b \in B$ either $\varphi(M, b) \supseteq \psi(M, b_1, ..., b_n)$ or $\varphi(M, b) \cap \psi(M, b_1, ..., b_n) = \emptyset$).

Distal case

Fact

[CS18b, Corollary 4.8.] Let $\mathcal{M}$ be a distal structure and let a formula $\phi(x, y, z)$ be given. Then there is some $\delta = \delta(\phi) > 0$ and formulas $\psi_1(x, z_1)$ and $\psi_2(y, z_2)$ depending just on ϕ and satisfying the following. For any definable relation $R(x, y) = \phi(x, y, c)$ for some $c \in M^{|z|}$ and finite $A \subseteq M^{|x|}, B \subseteq M^{|y|}$ there are some $A' \subseteq A, B' \subseteq B$ with $|A'| \geq \delta|A|, |B'| \geq \delta|B|$ and
(1) the pair A', B' is R-homogeneous,
(2) there are some $c_1 \in A^{|z_1|}$ and $c_2 \in B^{|z_2|}$ such that $A' = \psi_1(A, c_1)$ and $B' = \psi_2(B, c_2)$.

$ACVF_{0,0}$ case

A corollary is strong Erdős-Hajnal property in $ACVF_{0,0}$ (algebraically closed field of characteristic zero whose residue field also has characteristic zero):

Fact

[CS18b, Example 4.11. (2)] Let $\mathcal{M} \models ACVF_{0,0}$ and let a formula $\varphi(x, y, \bar{z})$ be given. Then there is some $\delta = \delta(\varphi) > 0$ such that for any definable relation $E(x, y) = \varphi(x, y, \bar{c})$ for some $\bar{c} \in M^{|\bar{z}|}$ and finite disjoint $X \subseteq M$, $Y \subseteq M$, there are some $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq \delta|X|$, $|Y'| \geq \delta|Y|$ and $X' \times Y' \subseteq E$ or $X' \times Y' \subseteq \neg E$.

$ACVF_{p,q}$ in general

(F.) Let $\mathcal{M} \models ACVF_{p,q}$ and let a formula $\varphi(x, y, \bar{z})$ be given. Then there is some $\delta = \delta(\varphi) > 0$ such that for any definable relation $E(x, y) = \varphi(x, y, \bar{c})$ for some $\bar{c} \in M^{|\bar{z}|}$ and finite disjoint $X \subseteq M$, $Y \subseteq M$, there are some $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq \delta|X|$, $|Y'| \geq \delta|Y|$ and $X' \times Y' \subseteq E$ or $X' \times Y' \subseteq \neg E$.

Bounded VC-minimal complexity case

Definition

Given a set U, a family of subsets $\Psi = \{B_i : i \in I\} \subseteq \mathcal{P}(U)$, where I is some index set, is called a *directed family* if for any $B_i, B_j \in \Psi$, $B_i \subseteq B_j$ or $B_j \subseteq B_i$ or $B_i \cap B_j = \emptyset$.

Definition

Given a directed family Ψ of subsets of U, a set $B \in \Psi$ is a called a Ψ-*ball*. A set $S \subseteq U$ is a Ψ-*Swiss cheese* if $S = B \setminus (B_0 \cup \ldots \cup B_n)$, where each of $B, B_0, \ldots, B_n$ is a Ψ-ball. We will call B an *outer ball* of S, and each B_i is called a *hole* of S.

Bounded VC-minimal complexity case

Definition

Given a finite bipartite graph $(X, Y; E)$, we say it has *VC-minimal complexity* $< N$ if there is a directed family Ψ of subsets of Y such that for each $a \in X$, $E(a, Y)$ is a finite disjoint union of Ψ-Swiss cheeses and the number of outer balls + the number of holes $< N$. i.e.

if $E(a, Y) = (B_{11} \setminus (B_{12} \cup ... \cup B_{1d(1)})) \dot\cup ... \dot\cup (B_{s1} \setminus (B_{s2} \cup ... \cup B_{sd(s)}))$,

then $d(1) + ... + d(s) < N$.

Bounded VC-minimal complexity case

Theorem

(F.) For $N > 0$, let $k_N = \dfrac{1}{2^{N+4}}$. If a finite bipartite graph $(X, Y; E)$ has VC-minimal complexity $< N$ then there exist $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq k_N|X|$, $|Y'| \geq k_N|Y|$ such that $X' \times Y' \subseteq E$ or $X' \times Y' \cap E = \emptyset$.

Strong Erdős-Hajnal

Question

For which family of graphs does strong Erdős-Hajnal property hold?

Strong Erdős-Hajnal

In [Chu+20], Chudnovsky, Scott, Seymour and Spirkl characterized the families of graphs that are defined by omitting a finite family of graphs and have strong Erdős-Hajnal.

Fact

[Chu+20] For every forest H, there exists $\epsilon > 0$ such that for every graph G with $|G| > 1$ that is both H-free and $\overline{H}$-free, there is a pair (A, B) with $|A|, |B| \geq \epsilon |G|$ such that $A \times B \subseteq E$ or $A \times B \cap E = \emptyset$.

Bounded VC-minimal complexity case

The bounded VC-minaimal complexity case is different:

Remark

The family of forests can be shown to have VC-minimal complexity ≤ 2 and thus the VC-minimal case is not covered in [Chu+20].

For a forest H, $v \in V(H)$, let $B_{v,\triangleleft}$ denote the set of the predecessor of v and $B_{v,\triangleright}$ denote the set of successors of v. Consider the family $\mathcal{F}_H := \{B_{v,\triangleleft}, B_{v,\triangleright} : v \in H\}$. $\mathcal{F}_H$ is directed: Let $v, w \in V(H)$. If $B_{v,\triangleleft} \cap B_{w,\triangleright} \neq \emptyset$, then since $B_{v,\triangleleft}$ is a singleton, $B_{v,\triangleleft} \subseteq B_{w,\triangleright}$. Similarly, if $B_{v,\triangleleft} \cap B_{w,\triangleleft} \neq \emptyset$, then $B_{v,\triangleleft} \subseteq B_{w,\triangleleft}$. If $B_{v,\triangleright} \cap B_{w,\triangleright} \neq \emptyset$, then $v = w$ and $B_{v,\triangleright} = B_{w,\triangleright}$. For any $v \in V(H)$, $E_v = B_{v,\triangleleft} \sqcup B_{v,\triangleright}$. So given any forest $H = (V(H), E)$ and disjoint $X, Y \subseteq V(H)$, the bipartite graph $(X, Y; E)$ has VC-minimal complexity ≤ 2.

Comb lemma

Lemma

(F.) Given $k \in \mathbb{N}, d \in \mathbb{R}$ *with* $k \geq 2$, $d \geq 2$, *there exists* $\tau_0 = \tau_0(k, d)$, $L_0 = L_0(k, d)$ *satisfying the following:*
Let $\tau < \tau_0$, G *a strongly* $\binom{k}{2}$*-free* τ*-critical graph, and* $\mathcal{A} = (A_i : 1 \leq i \leq t) \subseteq G$ *an equicardinal blockade of width* $\frac{|G|}{t^d}$ *with* $\frac{|G|}{t^{2d}} \leq W_G$, *of length* $L_0 \leq t \leq 2|G|^{\frac{1}{d}}$ *such that for all* $a \in A$, $|E(a, A)| < \frac{|G|}{t^d}$.
Then there exist $b \in A$, *an* $(t', \frac{|G|}{t^{2d+2}})$*-comb* $((a_j, A'_j) : 1 \leq j \leq t')$ *in* $(E_b \cap A, \neg E_b \cap A)$ *such that* $\mathcal{A}' = (A'_j : 1 \leq j \leq t')$ *is an equicardinal minor of* $\mathcal{A}$ *with width* $\geq \frac{|G|}{t^{2d+2}}$, *length* $\geq t^{\frac{1}{8}}$.

Progress in combinatorics

Definition

For $\epsilon > 0$, we say that G is *ϵ-sparse* if it has maximum degree $\leq \epsilon|G|$, and *ϵ-restricted* if G or $\overline{G}$ is ϵ-sparse.

Definition

A hereditary class $\mathcal{C}$ has the *polynomial Rödl property* if there exists $C > 0$ such that for every $\epsilon \in (0, \frac{1}{2})$, every graph $G \in \mathcal{C}$ contains an ϵ-restricted induced subgraph on at least $\epsilon^C|G|$ vertices.

Conjecture

[FS08](Fox–Sudakov) For every graph H, there exists $d > 0$ such that for every $\epsilon \in (0, 1/2)$, and every H-free graph G, there is an ϵ-restricted subset of $V(G)$ with size at least $\epsilon^d|G|$.

Progress in conbinatorics

In [NSS23], Nguyen, Scott, and Seymour showed that polynomial Rödl property holds for graphs with bounded VC-dimension. Hence Erdős-Hajnal property holds for graphs with bounded VC-dimension (NIP graphs).

Fact

[NSS23, Theorem 1.5.] For every $d \geq 1$*, the class of graphs of VC-dimension at most* d *has the polynomial Rödl property.*

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