Enumeration degrees and applications

Keng Meng Ng

Nanyang Technological University, Singapore

UW-Madison
Logic Seminar
21 May 2024

Introduction

Motivating questions

Study how computation interacts with mathematical concepts.

Complexity of constructions and objects we use in mathematics

A large focus of computability theory is to calibrate the notion of *relative computational strength*

The most common way of doing this is through the use of *Turing reducibility* $\leq_T$.

The Turing reducibility measures relative algorithmic content (both positive and negative information) of two objects. Useful if information is presented or understood as "decision procedure".

Motivating questions

However in some cases it may be useful to be able to compare the relative *positive* algorithmic content between two objects.

The *enumeration reducibility* captures this by allowing only positive information about one set to produce positive information about another set.

As usual this pre-ordering induces a degree structure, the enumeration degrees $\mathcal{D}_e$.

Motivation for studying the structure of the enumeration degrees comes from:

- Connections with the Turing degrees
- Applications to computable topology and analysis

The enumeration degrees

Definition (1950s)

We say that $A \leq_e B$ if there is a c.e. set W such that for every n,

$$n \in A \Leftrightarrow D_e \subseteq B \text{ for some } \langle n, e \rangle \in W.$$

Here D_e is the canonical index of a finite set.

This only compares positive information between A and B.

This is different from A is c.e. relative to B, which is equivalent to $A \leq_e B \oplus B^c$.

The enumeration degrees

An equivalent formulation of $A \leq_e B$ is that every enumeration of B (as an element of ω^ω) computes an enumeration of A. By a result of Selman, this can be made uniform.

For example, if S is any set of vertices in a computable graph, then $N(S) \leq_e S$, where $N(S)$ is the set of neighbours of S.

Therefore $S^c \leq_e S$ if S is a maximal independent set of vertices (in a computable graph). The converse is not true unless S is c.e.

The enumeration degrees

The degree structure $\mathcal{D}_e$ induced by $\leq_e$ consists of the *enumeration degrees*. It is an upper semi-lattice (under the usual $\oplus$) with the least element 0_e.

Connections with $\leq_T$:

$$A \leq_T B \quad \Leftrightarrow \quad A \oplus A^c \text{ is c.e. relative to } B \quad \Leftrightarrow \quad A \oplus A^c \leq_e B \oplus B^c.$$

Therefore, the Turing degrees embed naturally into $\mathcal{D}_e$ via:

$$deg_T(A) \mapsto deg_e(A \oplus A^c),$$

preserving order, $\vee$ and the jump. The range of this embedding are the *total* enumeration degrees.
We want to study the structure of the (local) enumeration degrees.

Definability in $\mathcal{D}_e$

Theorem (Shore, Slaman)

The Turing jump is first order definable.

In $\mathcal{D}_e$, the *enumeration jump* is defined by $A' = K_A \oplus K_A^c$, where $K_A = \bigoplus_{e \in \omega} \Phi_e(A)$. As usual we have $A <_e A'$.

Note that $A \equiv_e K_A$ and $K_A \not\leq_e K_A^c$ in general.

Theorem (Kalimullin, 2003, Ganchev, Soskova 2012)

The enumeration jump is first order definable.

Theorem (Ganchev, Soskova 2015, Cai, Ganchev, Lempp, Miller, Soskova 2016)

The total e-degrees are definable, both locally and globally.

Applications to Effective Topology

Motivating questions

We want to study computable uncountable structures. To apply tools of classical computability, we only consider structures that are *countably based*.

Computable analysis has laid the framework and provided intuition on working with "uncountable" effective objects.

Recall that a Polish space is a separable completely metrizable space.

Definition

A computable metric space (S, d) consists of a countable set $S = \{c_0, c_1, \cdots\}$ and $d : \mathbb{N}^2 \mapsto \mathbb{R}$ such that $d(c_i, c_j)$ is a computable real number, and $\overline{(S, d)}$ is Polish with metric induced by d.

Motivating questions

We wish to go a little further. What about countably based (not necessarily metrizable) topological spaces?

Definition

A topological space is effectively second countable if there is a countable base $\{B_i\}_{i\in\omega}$, and a computable function f such that $B_i \cap B_j = \bigcup_{k\in W_{f(i,j)}} B_k$, and where "$B_i \cap B_j \neq \emptyset$" is c.e.

These definitions allow one to study many effective aspects of non-countable countably based spaces.

Represented spaces

Definition

Let $(\mathcal{T}, \{B_n\}_{n\in\omega})$ be an effective second countable topological space. To do computability we identify a point $x \in \mathcal{T}$ with

$$\mathbf{NBase}(x) = \{e \in \omega \mid x \in B_e\}$$

Notice that the information for $x \in \mathcal{T}$ should be positively given. Hence, a name for x is any $p \in \omega^\omega$ such that $rng(p) = \mathbf{NBase}(x)$.

This gives a representation of $\mathcal{T}$, if $\mathcal{T}$ is T_0 (Kolmogorov).

The effective properties of $\mathbf{NBase}(x)$ depends on the presentation $\{B_n\}_{n\in\omega}$ of $\mathcal{T}$. We always fix the "standard copy".

$x \in \mathcal{T}$ is *computable* iff $\mathbf{NBase}(x)$ is c.e.

Using enumeration degrees

To each $x \in \mathcal{X}$ we assign the degree of difficulty of generating a name for x:

Definition (Kihara-Pauly, extending J. Miller)

Suppose $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Then

$$x \leq y \Leftrightarrow \mathbf{NBase}(x) \leq_e \mathbf{NBase}(y).$$

Conversely, given an enumeration degree $\boldsymbol{d}$, we say that $\boldsymbol{d}$ is an $\mathcal{X}$-degree if $\mathbf{NBase}(x) \in \boldsymbol{d}$ for some $x \in \mathcal{X}$.

Every second countable space $\mathcal{X}$ is associated with a set of enumeration degrees $\{deg_e(\mathbf{NBase}(x)) \mid x \in \mathcal{X}\}$.

This approach leads to a rich study of classical computability, computable analysis, and effective descriptive set theory.

Examples of second countable spaces

We can study

$$D_{\mathcal{X}} := \{deg_e(\mathbf{NBase}(x)) \mid x \in \mathcal{X}\} \subseteq \text{enumeration degrees}$$

for different second countable T_0 spaces $\mathcal{X}$.

Conversely, we also want to know which classes of enumeration degrees is a *spectrum*? I.e. equal to $D_{\mathcal{X}}$ for some second countable T_0 space $\mathcal{X}$?

The class of *all* enumeration degrees is a spectrum.

» Take $\mathbb{S}^\omega$ where $\mathbb{S}$ is the Sierpiński space $(\{0,1\}; \{\emptyset, \{1\}, \{0,1\}\})$ (T_0 space).

» Then for any $A \subseteq \omega$, $A \equiv_e \mathbf{NBase}_{\mathbb{S}^\omega}(\chi_A)$.

» Computable points in $\mathbb{S}^\omega$ are the c.e. sets.

Examples of second countable spaces

If $\mathcal{X} = 2^\omega, \omega^\omega$ or $\mathbb{R}^n$, then $D_\mathcal{X}$ = total enumeration degrees.

- For any $x \in 2^\omega$, $\mathbf{NBase}_{2^\omega}(x) = \{\sigma \in 2^{<\omega} \mid \sigma \subset x\}$. Thus, $\mathbf{NBase}_{2^\omega}(x) \geq_e \overline{\mathbf{NBase}_{2^\omega}(x)}$.
- Thus, $2^\omega, \omega^\omega$ and $\mathbb{R}^n$ correspond to the Turing degrees, and is the smallest spectrum.
- $x \in 2^\omega$ is a computable point iff x is computable.

Let $\mathbb{R}_<$ be the set $\mathbb{R}$ with the lower topology, with base $\{(q, \infty) \mid q \in \mathbb{Q}\}$. This is T_0 but not T_1.

- $x \in \mathbb{R}_<$ is a computable point iff x is a left-c.e. real.
- (Kihara-Pauly) $D_{\mathbb{R}_<}$ = the semirecursive e-degrees.

Continuous degrees

Definition (J. Miller)

An enumeration degree is continuous if it contains $\mathbf{NBase}_{\mathcal{M}}(x)$ for some point $x \in \mathcal{M}$ where $\mathcal{M}$ is some computable metric space.

Every total e-degree is continuous.

(J. Miller) Continuous degrees $= D_{C[0,1]} = D_{[0,1]^\omega}$.

(J. Miller) There is a non-total continuous degree.

(Andrews, Igusa, J. Miller, M. Soskova) An e-degree is continuous iff it is almost total ($\boldsymbol{a}$ is almost total if for every total $\boldsymbol{x} \not\leq \boldsymbol{a}$, $\boldsymbol{a} \cup \boldsymbol{x}$ is total).

Which classes of enumeration degrees are spectra?

Studying $D_{\mathcal{X}}$ for various second countable $\mathcal{X}$ leads to a zoo of classes of e-degrees.

A degree is cototal if it contains a set A such that $A \leq_e \overline{A}$.

All continuous degrees are cototal. In fact, they can be characterised as:

- (AGKLMSS) The degrees of complements of maximal independent sets in computable graphs.
- (J. Miller, M. Soskova) The degrees of sets with good approximations.
- (McCarthy) $D_{\mathcal{X}}$ where $\mathcal{X}$ is the maximal antichain space.
- (Kihara, N., Pauly) The collection of points in effective G_δ-spaces.

Which classes of enumeration degrees are spectra?

(Kihara, N, Pauly) Establishing further classifications of $D_{\mathcal{X}}$ and providing separations:

graph-cototal, telograph-cototal, Roy halfgraph-above, doubled co-*d*-CEA, Arens co-*d*-CEA, *n*-cylinder cototal, *n*-semirecursive, quasiminimality, etc.

(Jacobsen-Grocott) Provided further separations involving Roy halfgraph-above and Arens co-*d*-CEA degrees.

The structure of the enumeration degrees $\mathcal{D}_e$

Studying degree structures

Most degree structures (arising from classical computability) are very complicated partial orders.

Commonly studied degrees $\mathcal{D}_T, \mathcal{D}_T(\leq 0'), \mathcal{R}, \mathcal{D}_e, \mathcal{D}_e(\leq 0')$

Does the degree structure (as a partial order) have a decidable theory?
Usually the answer is "no", and in fact the theory is maximally complicated.

Well, in that case, is the existential theory decidable?
Usually the answer is "yes", because we can decide which finite partial orders are embeddable.

Usually, the $\exists\forall\exists$-theory can be shown to be undecidable.

Studying degree structures

The $\forall\exists$ theory is usually the difficult case

(Shore 78, Lerman 83) The $\forall\exists$ theory of $\mathcal{D}_T$ is decidable.
(Shore, Lerman 88) The $\forall\exists$ theory of $\mathcal{D}_T(\leq 0')$ is decidable.

The question remains open for $\mathcal{R}, \mathcal{D}_e, \mathcal{D}_e(\leq 0')$.

A main difficulty in the remaining cases is (downward) density:

- (Spector, Sacks) $\mathcal{D}_T$ and $\mathcal{D}_T(\leq 0')$ have minimal degrees.
- (Gutteridge) $\mathcal{D}_e$ does not have minimal degrees.
- (Sacks, Cooper) $\mathcal{R}$ and $\mathcal{D}_e(\leq 0')$ are dense partial orders.

The local enumeration degrees

We focus on the *local enumeration degrees* $\mathcal{D}_e(\leq 0')$, which are exactly the enumeration degrees of Σ^0_2 sets.

The $\exists\forall\exists$ theory is undecidable (Kent) and is computably isomorphic to the theory of first order arithmetic (Ganchev, Soskova).
The $\forall\exists$ theory of the local enumeration degrees remain open, but a well-known algebraic formulation is equivalent:

> *Given finite partial orders $P, Q_0, Q_1, \cdots, Q_n$ such that $P \subseteq Q_i$ for every $i \leq n$, does every embedding of P into $\mathcal{D}$ extend to an embedding of Q_i into $\mathcal{D}$ for some $i \leq n$?*

When $n = 0$ this is the *Extension of embeddings* problem.

We say that a configuration $P, Q_0, Q_1, \cdots, Q_n$ can be realised if every embedding of P extends to an embedding of some Q_i; otherwise we say that the configuration can be blocked.

The local enumeration degrees

(Lempp, Slaman and Sorbi) The extension of embeddings for $\mathcal{D}_e(\leq 0')$ is decidable.

Cooper showed that $\mathcal{D}_e(\leq 0')$ is dense (similar to the c.e. Turing degrees). Are they elementarily equivalent?

Theorem (Ahmad)

There are incomparable Σ^0_2 e-degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ such that if $\boldsymbol{x} < \boldsymbol{a}$ then $\boldsymbol{x} < \boldsymbol{b}$.

This became known as the so-called *Ahmad pair*. Denote this property as $\mathcal{A}(\boldsymbol{a}, \boldsymbol{b})$.

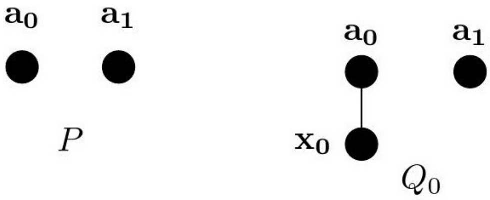
Note that if $\mathcal{A}(\boldsymbol{a}, \boldsymbol{b})$ then $\boldsymbol{a}$ is non-splittable, whereas every non-zero c.e. degree is splittable (Sacks Splitting).

The local enumeration degrees

Analyzing the $\forall\exists$ theory for $\mathcal{D}_e(\leq 0')$ seems to involve the following basic obstacles:

There are incomparable degrees $\boldsymbol{a}, \boldsymbol{b}$ such that $\mathcal{A}(\boldsymbol{a}, \boldsymbol{b})$.
For every $\boldsymbol{x}$, if $\boldsymbol{x} < \boldsymbol{a}$ then $\boldsymbol{x} < \boldsymbol{b}$.

This means that the following configuration is blockable:



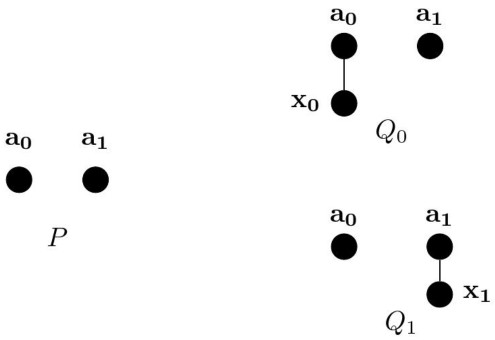
This is already covered by the extension of embeddings of Lempp, Slaman and Sorbi.

The local enumeration degrees

Theorem (Ahmad, Lachlan)

There is no symmetric Ahmad pair, i.e. if $\mathcal{A}(\boldsymbol{a}, \boldsymbol{b})$ and $\mathcal{A}(\boldsymbol{b}, \boldsymbol{a})$ then $\boldsymbol{a} = \boldsymbol{b}$.

This means that the following configuration is realisable:



The local enumeration degrees

These led us to consider the following subproblem of deciding all $\forall\exists$ sentences:

Problem

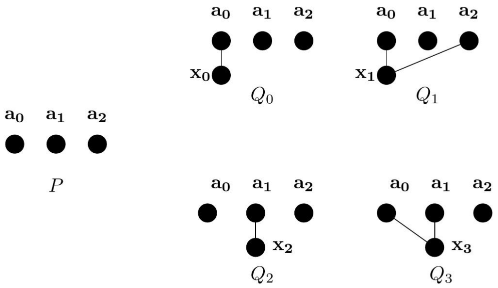
We restrict P to a finite antichain of degrees, and each Q_i to be a single point extension of P such that every new element is below at least one (possibly more) elements of P.

Other variations on *one point extensions* can be decided easily:

- P is a finite chain of degrees and Q_i are one point extensions of P.
- P is a finite antichain of degrees and at least one Q_i adds one point incomparable to P
- P is a finite antichain of degrees and every Q_i is a one point extension adding elements above P

$|P| = 3$

Under this formulation, all cases when $|P| = 2$ can be decided. The next configuration to consider is:



This configuration can be blocked if the natural extension of an Ahmad pair to an Ahmad triple holds, i.e. incomparable Σ^0_2 degrees $\boldsymbol{a}_0, \boldsymbol{a}_1$ and $\boldsymbol{a}_2$ such that $\mathcal{A}(\boldsymbol{a}_0, \boldsymbol{a}_1)$ and $\mathcal{A}(\boldsymbol{a}_1, \boldsymbol{a}_2)$.

$|P| = 3$

Theorem (Goh, Lempp, N, Soskova)

There is no Ahmad triple of Σ^0_2 degrees.

To prove this we first find a direct proof of "no symmetric Ahmad pair". (Ahmad, Lachlan's proof was an indirect one using Gutteridge operators).

(Ahmad, Lachlan) There is no symmetric Ahmad pair, i.e. if $\mathcal{A}(\boldsymbol{a}, \boldsymbol{b})$ and $\mathcal{A}(\boldsymbol{b}, \boldsymbol{a})$ then $\boldsymbol{a} = \boldsymbol{b}$.

It turns out that the only property needed to make it work was that the same degree $\boldsymbol{b}$ cannot simultaneously be the left half and the right half of an Ahmad pair.

$|P| = 3$

Theorem (Goh, Lempp, N, Soskova)

If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are incomparable, then we cannot have $\mathcal{A}(\boldsymbol{a}, \boldsymbol{b})$ and $\mathcal{A}(\boldsymbol{b}, \boldsymbol{c})$.

Proof.

First we build $\boldsymbol{x}_0 < \boldsymbol{a}$.

If we fail to make $\boldsymbol{x}_0 \not\leq \boldsymbol{b}$, then we begin building another $\boldsymbol{x}_1 < \boldsymbol{b}$.

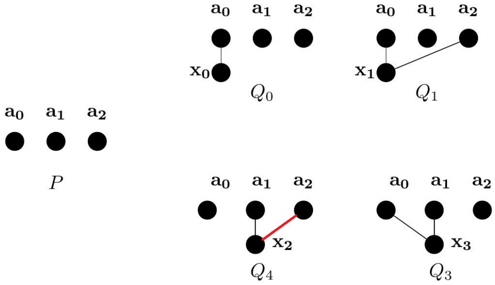
If we fail to make $\boldsymbol{x}_1 \not\leq \boldsymbol{c}$, then we begin building another $\boldsymbol{x}_2 < \boldsymbol{a}$. This time we will succeed in making $\boldsymbol{x}_2 \not\leq \boldsymbol{b}$. □

$|P| = 3$

Theorem (Goh, Lempp, N, Soskova)

There is no Ahmad triple of Σ^0_2 degrees.

Replacing Q_2 with Q_4 gives a completely different picture:



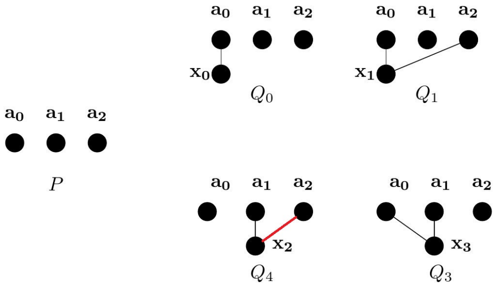
How to block this configuration?

$|P| = 3$

Theorem (Goh, Lempp, N, Soskova)

There are incomparable Σ^0_2 degrees $\boldsymbol{a}_0, \boldsymbol{a}_1$ and $\boldsymbol{a}_2$ such that $\mathcal{A}(\boldsymbol{a}_0, \boldsymbol{a}_1)$ and $\mathcal{A}(\boldsymbol{a}_0, \boldsymbol{a}_2)$, and for every $\boldsymbol{x}$, if $\boldsymbol{x} < \boldsymbol{a}_1, \boldsymbol{a}_2$ then $\boldsymbol{x} < \boldsymbol{a}_0$.

This blocks the configuration:

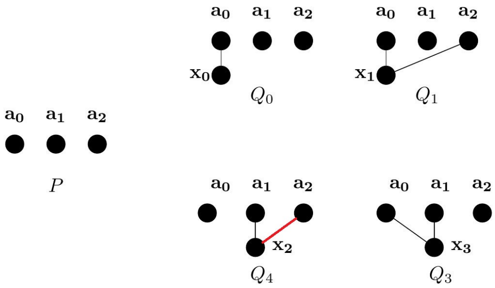


Difficulty: By the fact that there are no symmetric Ahmad pairs, neither $\mathcal{A}(\boldsymbol{a}_1, \boldsymbol{a}_0)$ nor $\mathcal{A}(\boldsymbol{a}_2, \boldsymbol{a}_0)$ can hold, so the condition "$\boldsymbol{x} < \boldsymbol{a}_1, \boldsymbol{a}_2$" is absolutely necessary.

$|P| = 3$

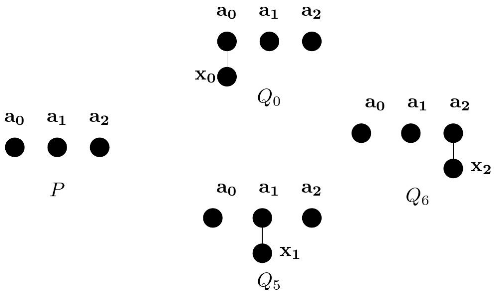
There is another technique used in the analysis: mixing in minimal pair requirements.
The previous theorem is such an example.

- To block P, Q_1, Q_3, Q_4, we can make $\boldsymbol{a}_0, \boldsymbol{a}_1$ and $\boldsymbol{a}_2$ form pairwise minimal pairs.
- To block P, Q_0, Q_4, we can make $\mathcal{A}(\boldsymbol{a}_0, \boldsymbol{a}_1)$ and $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 = 0$.



$|P| = 3$

We consider the next configuration:



To block this (if possible), we cannot simply rely on constructing Ahmad pairs, since if we make $\mathcal{A}(\boldsymbol{a}_0, \boldsymbol{a}_1)$, then we cannot make $\mathcal{A}(\boldsymbol{a}_1, \boldsymbol{a}_2)$ or $\mathcal{A}(\boldsymbol{a}_1, \boldsymbol{a}_0)$ true.

It's clear that we need a (new) method to build $\boldsymbol{a}_0, \boldsymbol{a}_1$ and $\boldsymbol{a}_2$ such that for every $\boldsymbol{x} < \boldsymbol{a}_0$, we have $\boldsymbol{x} < \boldsymbol{a}_1$ or $\boldsymbol{x} < \boldsymbol{a}_2$.

$|P| = 3$

Definition (Generalized Ahmad pair)

If F is a finite set of degrees, and $\boldsymbol{a} \not\leq \boldsymbol{b}$ for any $\boldsymbol{b} \in F$, we say that $\mathcal{A}(\boldsymbol{a}, F)$ if for every $\boldsymbol{x} < \boldsymbol{a}$ we have $\boldsymbol{x} < \boldsymbol{b}$ for some $\boldsymbol{b} \in F$.

Theorem (Goh, Lempp, N, Soskova)

There are incomparable Σ^0_2 degrees $\boldsymbol{a}_0, \boldsymbol{a}_1$ and $\boldsymbol{a}_2$ such that $\mathcal{A}(\boldsymbol{a}_0, \{\boldsymbol{a}_1, \boldsymbol{a}_2\})$, but neither $\mathcal{A}(\boldsymbol{a}_0, \boldsymbol{a}_1)$ nor $\mathcal{A}(\boldsymbol{a}_0, \boldsymbol{a}_2)$ holds.

The difficulty is to implement the switching of outcomes from coding $\boldsymbol{x} < \boldsymbol{a}_0$ into $\boldsymbol{a}_1$ to coding $\boldsymbol{x}$ into $\boldsymbol{a}_2$.

$|P| = 3$

Theorem (Gutteridge)

There is an operator G such that for any set A, $G(A) \leq_e A$. If A is not c.e. then $G(A) <_e A$, and if A is not Δ_2^0 then $G(A)$ is not c.e.

The Gutteridge operator has been a key tool in our analysis. (Used in Ahmad, Lachlan's proof of no symmetric Ahmad pair).

- Each column $G(X)^{[n]}$ is finite.
- Each $n \in X$ is encoded in $G(X)^{[n]}$, and the location of this can be computed by $\emptyset'$.

Using this, we can show:

Theorem (Goh, Lempp, N, Soskova)

There are no incomparable Σ_2^0 degrees $\boldsymbol{a}_0, \cdots, \boldsymbol{a_n}$ such that $\bigwedge_{i \leq n} \mathcal{A}(\boldsymbol{a_i}, F - \{\boldsymbol{a_i}\})$ where $F = \{\boldsymbol{a}_0, \cdots, \boldsymbol{a_n}\}$.

Thus the configuration P, Q_0, Q_5, Q_6 turns out to be realisable.

Where is this all going?

It turns out that the previous analysis of different cases yields all the tools needed.

Theorem (Goh, Lempp, N, Soskova, in preparation)

There are no finite sets of degrees $F = \{\boldsymbol{a}_1, \cdots, \boldsymbol{a_n}\}, G_1, \cdots, G_n$ *and a degree* $\boldsymbol{a}_0$ *such that all degrees are incomparable, and where*

$$\mathcal{A}(\boldsymbol{a}_0, F) \wedge \bigwedge_{0<i\leq n} \mathcal{A}(\boldsymbol{a_i}, G_i)$$

This simultaneously generalises all "negative" results that produce realisable configurations in the previous analysis.

The proof uses ideas from the previous theorem on "no Ahmad triples", using a modular approach.

The criteria for one point extensions

This allows us to formulate a condition for which configurations can be blocked:

Theorem (Goh, Lempp, N, Soskova, in preparation)

Let $\mathcal{S} \subseteq \mathcal{P}(\{0, \cdots, n\}) - \{\emptyset\}$. (Here, each $S \in \mathcal{S}$ represents a one point extension Q_S of $P = \{\boldsymbol{a}_0, \cdots, \boldsymbol{a_n}\}$).
Let $S_0 = \{i \leq n \mid \{i\} \in \mathcal{S}\}$ and $S_1 = \{0, \cdots, n\} - S_0$. Then the configuration $P, (Q_S)_{S \in \mathcal{S}}$ can be blocked if and only if (omitting trivial cases) there is an assignment $\nu : S_0 \to \mathcal{P}(S_1) - \{\emptyset\}$ satisfying:

- *For each $i \in S_0$, $\{i\} \cup \nu(i) \notin \mathcal{S}$, and*
- *for each $F \subseteq S_0$ with $|F| > 1$ we have $\bigcap\{\nu(i) \mid i \in F\} \notin \mathcal{S}$.*

Next steps

The next step towards deciding the $\forall\exists$ theory of $\mathcal{D}_e(\leq 0')$ is to obtain a condition for one point extensions both above and below elements of P.

Theorem (Kalimullin, Lempp, N., Yamaleev)

There are no incomparable Σ^0_2 degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a} \cup \boldsymbol{b} = 0'$ and $\mathcal{A}(\boldsymbol{a}, \boldsymbol{b})$.

The proof is a non-uniform finite injury argument.

This shows that $P = \{\boldsymbol{a}, \boldsymbol{b}\}$, $Q_0 = \{\boldsymbol{x} < \boldsymbol{a}, \boldsymbol{b}\}$, $Q_1 = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{a}, \boldsymbol{x} \geq \boldsymbol{b}\}$ is realisable.

Both questions (Existence of an Ahmad triple and a cupping Ahmad pair) were asked by Kent (2007).

Thank you!