Beyond the ceers

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After a full semester on the ceers, you (*you*, the audience) are now among the world's experts on the topic—congratulations!

Yet, computable reducibility ($\leq_c$) makes perfect sense for equivalence relations on ω of *any* complexity. Today, I'm going to review a couple of topics dealing with arithmetical equivalence relations.

Universality

Introducing universality

For a complexity class $\mathcal{C}$, an equivalence relation $E \in \mathcal{C}$ is universal if every equivalence relation in $\mathcal{C}$ reduces to E.

Universal ceers have been studied since the 1980s. The **Cantor-Schröder-Bernstein** property fails for ceers: there are universal ceers that, although bi-reducible, are not computably isomorphic. This discovery initiated the long-standing program of classifying isomorphism types of universal ceers via natural combinatorial properties.

Climbing up the arithmetical hierarchy, new phenomena occur. Let's begin with a curious observation. Denote by $=^{ce}$ the equality (of indices) of c.e. sets.

Proposition (Andrews, S.)

$=^{ce}$ is not Π^0_2 universal. In fact, there is a d-c.e. equivalence relation E that doesn't reduce to it.

The proof, although simple, distillates the two-dimensional character of computable reducibility, which distingues it from classic reducibilities, such as Turing or *m*-reducibility.

Proof idea: To diagonalize against a potential reduction φ_e, let $e_0 \not\mathrel{E} e_1$. We use fresh numbers e_z's, with $z > 1$, to gradually copy $W_{\varphi_e(e_0)}$ into $W_{\varphi_e(e_1)}$ and vice versa: e.g., whenever $v \searrow W_{\varphi_e(e_0)}$ we let $e_0 \mathrel{E} e_z$, for a fresh z; next, if $v \searrow W_{\varphi_e(e_z)}$, we let $e_0 \not\mathrel{E} e_z$ and $e_z \mathrel{E} e_1$.

Universal Σ_n equivalence relations

The existence of a universal element for a class $\mathcal{C}$ of equivalence relations is readily settled if $\mathcal{C}$ can be effectively enumerated: just take $\bigoplus_{e\in\omega} R_e$, where $(R_e)_{e\in\omega}$ is an effective enumeration of $\mathcal{C}$.

Now, the reflexive and symmetric closure of a binary relation doesn't increase the complexity of the relation, while the transitive closure of a binary relation S is c.e. in S. Thus, for all n, there is an equivalence relation which Σ^0_n universal.

The case of Π-levels, as suggested by the nonuniversality of $=^{ce}$, is more delicate.

No effective enumeration of all Π^0_1 equivalence relations

Proposition (Ng, Yu)

There is no effective enumeration of all co-ceers.

Proof idea: Suppose that $(R_e)_{e\in\omega}$ is an effective enumeration of all co-ceers. We build E so that the ith column $E^{[i]}$ of E witness that $E \neq R_i$. To do so, let $\langle e, 0\rangle \not{E} \langle e, 1\rangle$. Then, we let $\langle e, s+2\rangle$ be E-inequivalent with both $\langle e, 0\rangle$ and $\langle e, 1\rangle$; unless at stage s we witness that $\langle e, 0\rangle \not{R_i} \langle e, 1\rangle$, in which case we keep

$$\langle e, 0\rangle \, E \, \langle e, s+2\rangle \, E \, \langle e, 1\rangle.$$

By transitivity, the opponent must show us that $\langle e, s+2\rangle$ is R_i-inequivalent to $\langle e, 0\rangle$ or $\langle e, 1\rangle$: this gives an immediate way to diagonalize while ensuring that E satisfies transitivity.

There is a universal co-ceer

However, a class $\mathcal{C}$ may have universal member even if $\mathcal{C}$ cannot be effectively enumerated.

Theorem (Ng, Yu)

There is an effective enumeration $(E_e)_{e\in\omega}$ *of co-ceers so that every co-ceer* S *reduces to some* E_e.

It immediately follows that there is a universal co-ceer. Denote its degree by π.

Interestingly, the existence of an effective enumeration of all degrees of co-ceers seems to be an open problem.

Realizing π

π has natural realizations.

Recall that (unary) quadratic time functions over the alphabet $\{0, 1\}$ are represented by Turing machines equipped with a counter forcing them to stop in time $O(n^2)$, where n is the length of the input.

Theorem (Ianovski, R. Miller, Ng, Nies)

Equality of quadratic time computable functions is a universal co-ceer.

Can we do better? Uri and I believe so.

Realizing π, continued

Andrews, S.: *Equality of linear time computable functions is a universal co-ceer (in fact, the same holds for any time bound you wish).*

Proof idea: Given a co-ceer E, we shall construct a family of linear time computable functions (f_e) so that $u\,E\,v$ iff $f_u = f_v$. Say that a stage s is z-consistent, if $E \upharpoonright_{\leq z}$ satisfies transitivity. The key idea is to meet the following requirements:

1. Each f_z is defined at a z-consistent stage, by first prescribing that f_z copies f_x for the least x in $[z]_E$;
2. Whenever we witness a failure of transitivity on the functions f_z's currently defined, we restrain from defining new ones and we extend each f_z as previously prescribed until consistency is restored. At that point, we update the behavior of such functions by mimicking that of E.

On the other hand, universality disappears at the next Π-level of the arithmetical hierarchy.

Theorem (Ianovski, R. Miller, Ng, Nies)

There is no universal Π^0_n equivalence relation, for $n > 1$.

The result seems in sharp constrast with the existence of universal co-ceers. However, it can be explained by observing that the last construction (in fact, a simplified version of it) shows that each co-ceer S can be encoded into the columns of a suitable computable set Y_S:

$$i \, S \, j \Leftrightarrow Y_S^{[i]} = Y_S^{[j]}, \tag{†}$$

and of course this equation relativizes.

In particular, we have that each Π^0_2 equivalence relation S can be encoded into the columns of a Δ^0_2 equivalence relation Y_S.

But then, it's clear that no Π^0_2 equivalence relation can be universal. Suppose otherwise, and this witnessed by S. Then Y_S, being Δ^0_2, must be α-c.e. for some α. Thus, S cannot be above, say, the equality (of indices) of $\alpha+1$-c.e. sets ($=^{\alpha+1\text{-c.e.}}$).

It suffices to relativize again to deduce that there is no Π^0_{n+2} which is universal for $\mathbf{0}^{(n)}$ reducibility—let alone computable reducibility.

Moral: co-ceers are stange. Feel free to explore them!

Effectivizing the Borel theory

Effectivizing benchmark relations, I

Following Coskey, Hamkins, and R. Miller (2012), we adapt benchmark relations from the Borel theory by restricting them to the c.e. sets. This naturally give rise to equivalence relations on the natural numbers. Indeed, if E is on the c.e. sets, then we let, for all $e, i \in \omega$,

$$e \, E^{ce} \, i \Leftrightarrow W_e \, E \, W_i.$$

So, $\mathsf{Id}(2^\omega)$ translates to the equality of c.e. sets, given by

$$e =^{ce} i \Leftrightarrow W_e = W_i.$$

Similarly, we let

$$e \, E_0^{ce} \, i \Leftrightarrow W_e \triangle W_i \text{ is finite.}$$

E_1^{ce} is defined by regarding at c.e. sets as subsets of $\omega \times \omega$:

$$e \, E_1^{ce} \, i \Leftrightarrow (\forall^\infty n)(W_e^{[n]} = W_i^{[n]}).$$

Reductions between E^{ce}'s, I

Theorem (Coskey, Hamkins, R. Miller)

$\mathrm{Id}(\omega) <_c =^{ce} <_c E_0^{ce}$.

Proof idea: The reductions closely resemble the Borel ones. Nonreductions are far easier to get than in the Borel framework. Calculating the complexity of the relations involved (as set of pairs) suffices:

- $\mathrm{Id}(\omega)$ is Δ^0_1,
- $=^{ce}$ is Π^0_2,
- E_0^{ce} is Σ^0_3.

Reductions between E^{ce}'s, II

Theorem (Coskey, Hamkins, R. Miller)

$E_0^{ce} \sim_c E_1^{ce}$.

That E_1^{ce} reduces to E_0^{ce} is surprising and it breaks with the Borel theory. In fact, it turns out that E_0^{ce} is as complex as possible:

Theorem (Ianovski, R. Miller, Ng, Nies)

E_0^{ce} is Σ_3^0 universal.

Silver's dichotomy fails for computable reducibility

There is no analog of **Silver's dichotomy** for $\leq_c$. For all $e, i \in \omega$, define

- $e \, E_{min} \, i \Leftrightarrow (\min W_e = \min W_i)$,
- $e \, E_{max} \, i \Leftrightarrow (\max W_e = \max W_i$ or $|W_e| = |W_i| = \infty)$.

Theorem (Coskey, Hamkins, R. Miller)

E_{min} and E_{max} are c-incomparable and they both reduce to $=^{ce}$.

Other dichotomies fail as well (stay tuned). However, the failure of dichotomies is to be expected: first, contrary to $\leq_B$, computable reducibility is sensible to the complexity of relations/classes involved; secondly, controlling fixed points given by the recursion theorem is a formidable tool for diagonalizing.

Introducing cbers

A fundamental subclass of Borel equivalence relations, named countable Borel equivalence relations (cbers), consists of those with countable equivalence classes. This study is intertwined with that of the equivalence relations which can be realized by Borel actions of countable groups.

Let G be a group acting on a standard Borel space. Then the orbit equivalence relation E_G is given by

$$x \, E_G \, y \Leftrightarrow (\exists \gamma \in G)(\gamma \cdot x = y).$$

Group actions

For example,

- The action of $\mathbb{Z}$ on 2^ω induced by the odometer map (i.e., $+1 \bmod 2$ with right carry) produces an equivalence relation which almost coincides with E_0, but it glues $[1^\infty]_{E_0}$ with $[0^\infty]_{E_0}$.
- For each countable group G, the shift action of G on the space 2^G is given by

$$(g \cdot p)_h = p_{g^{-1}h},$$

for $g, h \in G$ and $p \in 2^G$. (If $G = \mathbb{Z}$, this corresponds to left shift of doubly-infinite binary sequences).

Theorem (Feldman, Moore)

If E is a cber on a standard Borel space X, then there is a countable group G and a Borel action of G on X such that $E = E_G$.

The proof relies on **Luzin-Novikov Uniformization**, which ensures that every countable Borel equivalence relation has a uniform Borel enumeration of each class.

The hierarchy of cbers is rich and complicated. However, it has a top element. Denote by E_∞ the shift action $\mathbb{F}_2$ (the free group with 2-generators) on $2^{\mathbb{F}_2}$.

Theorem (Dougherty, Jackson, Kechris)

E_∞ is a universal cber (that is, $E \leq_B E_\infty$ for all cbers E).

Orbit equivalence relations under computable lenses

Denote by **CE**, the collection of c.e. sets (to be understood *extensionally*, i.e., as just subsets of ω).

Coskey, Hamkins, R. Miller (2012):

- The action of a computable group G acting on **CE** is computable in indices if there is computable α so that

  $$W_{\alpha(g,e)} = g \cdot W_e.$$

  The induced orbit equivalence relation is denoted E^{ce}_G.
- E^{ce} is enumerable in indices if there is computable α so that, for all $i \in \omega$,

  $$e\, E^{ce}\, i \Leftrightarrow (\exists n)(W_{\alpha(e,n)} = W_i).$$

Realizing E_0^{ce} via a group action, I

Is there an effective analog of Feldman-Moore? That is, is it the case that any E^{ce} enumerable in indices is the orbit relation of an action computable in the indices? The answer is (again): no.

Theorem (Coskey, Hamkins, R. Miller)

E_0^{ce} is enumerable in indices but there is no group action G computable in the indices so that $E_0^{ce} = E_G^{ce}$.

One way to see this is by using the following lemma. Say that a given E_G^{ce} is permutation induced if there is a computable subgroup H of S_∞ so that

$$x \, E_G^{ce} \, y \Leftrightarrow (\exists \pi \in H)(W_y = \{\pi(n) : n \in W_x\}).$$

Realizing E_0^{ce} via a group action, II

Lemma (Andrews, S.)

Every orbit relation of a group action computable in indices is permutation induced.

So, when dealing with E_G^{ce}, we shall assume that G is a subgroup of S_∞ whose action on the c.e. sets is given, for all $\pi \in G$, by

$$\pi \cdot W_x = \{\pi(n) : n \in W_x\}.$$

From the lemma, it immediately follows that no E_G^{ce} glues c.e. sets of different size. So, e.g., neither E_0^{ce} nor E_1^{ce} can be realized by group actions computable in indices. To overcome this limitation, it is natural to relax the notion of realizability and reasoning up to $\leq_c$. Then, the next question arises:

Is there G so that $E_0^{ce} \sim_c E_G^{ce}$?

Realizing E_0^{ce} via a group action, III

Since E_0^{ce} is Σ_3^0 universal, then all E_G^{ce} reduce to it. So, the question is really whether E_0^{ce} can be encoded into some E_G^{ce}.

Let P be the subgroup of S_∞ generated by all permutations with finite support (i.e., those that move only finitely many elements).

Theorem (Andrews, S.)

$E_0^{ce} \sim_c E_P$.

The proof is a priority construction dealing with Σ_3^0 approximations. Yet, note that E_P is, in a sense, the closest you may get to E_0^{ce} by using permutations. Indeed, $i \, E_P \, j$ if and only if there is n so that:

- $|W_i \cap [0,n]| = |W_j \cap [0,n]|$ and $W_i \setminus [0,n] = W_j \setminus [0,n]$.

A new dichotomy

At this point, one may suspect that "few" orbit relations would be of the highest complexity (i.e., that of E_0^{ce}). This is not the case.

In fact, we have obtained the following neat – and quite unexpected – dichotomy:

Theorem (Andrews, S.)

For all groups G acting computably in indices (and faithfully),

- *If G is finite, then* $E_G^{ce} \sim_c =^{ce}$,
- *If G is infinite, then* $E_G^{ce} \sim_c E_0^{ce}$.

Hence, E_∞^{ce} has many natural realizations.

Failures of Feldman-Moore and Glimm-Effros

Anyway, the analog of **Feldman-Moore** theorem fails also working up to $\leq_c$, e.g., E_{min} and E_{max} are enumerable indices but, being stricly below $=^{ce}$, they cannot be equivalent to any E_G^{ce}. In fact,

Theorem (Andrews, S.)

1. *There is an infinite chain of equivalence relations which are enumerable in the indices between* $=^{ce}$ *and* E_0^{ce}.
2. *There is an infinite antichain of equivalence relations enumerable in indices between* $=^{ce}$ *and* E_0^{ce}.

Thus, there is no computable analog of **Glimm-Effros**.

Thank you!

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