Countable Ordered Groups and Weihrauch Reducibility

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Outline

Reverse mathematics

- Reverse mathematics study the strength of axioms that is needed to prove theorems of ordinary mathematics over a weak base theory.
- It is usually studied using subsystems of second order arithmetic.
- In the appropriate base theory, we can code well-orders, groups etc in $\mathbb{N}$. Codings can be different. We say it is an ω-presentation or ω-copy for one coding.

Big five

1. RCA_0: $\mathsf{PA}^- + \mathsf{I}\Sigma^0_1 + \Delta^0_1$-$\mathsf{CA}$
2. WKL_0: RCA_0 + some form of weak könig lemma
3. ACA_0: RCA_0 + arithmetical comprehension axiom
4. ATR_0: ACA_0 + arithmetical transfinite recursion scheme
5. Π^1_1-CA_0: RCA_0 + Π^1_1-comprehension axiom

Theorem

The following are equivalent over RCA_0*:*

1. Π^1_1-CA_0
2. *For any sequence of trees* $\langle T_k : k \in \mathbb{N}\rangle$, $T_k \subseteq \mathbb{N}^{<\mathbb{N}}$, *there exists a set* X *such that* $\forall k(k \in X \leftrightarrow T_k$ *has a path*$)$.

Order Type of Countable Ordered Group

Theorem (Maltsev, 1949)

The order type of a countable ordered group is $\mathbb{Z}^{\alpha}\mathbb{Q}^{\varepsilon}$, where α is an ordinal and $\varepsilon = 0$ or 1.

Definition

An ordered group is a pair $(G, \leq_G)$, where G is a group, $\leq_G$ is a linear order on G, and for all $a, b, g \in G$, if $a \leq b$ then $ag \leq bg$ and $ga \leq gb$.

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Definition

Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be linear orders. The product XY is the linear order $(Z, \leq_Z)$ where

$$Z = \{\langle x, y\rangle : x \in X \wedge y \in Y\},$$

$$\langle x_1, y_1\rangle \leq_Z \langle x_2, y_2\rangle \leftrightarrow y_1 <_Y y_2 \vee (y_1 = y_2 \wedge x_1 \leq_X x_2).$$

$\mathbb{Z}^X$ is the set of functions $f : X \to \mathbb{Z}$ with finite support. If $f \neq g$, then $f < g$ if and only if $f(x) <_{\mathbb{Z}} g(x)$ where x is the maximum value of X on which f and g disagree.

Order Type of Countable Ordered Group

Theorem (Solomon, 2001)

The following are equivalent under RCA_0*:*

1. Π^1_1-CA_0
2. *Let G be a countable ordered group. There is a well-order α and $\varepsilon \in \{0,1\}$ such that $\mathbb{Z}^\alpha \mathbb{Q}^\varepsilon$ is the order type of G.*
3. *Let G be a countable abelian ordered group. There is a well-order α and $\varepsilon \in \{0,1\}$ such that $\mathbb{Z}^\alpha \mathbb{Q}^\varepsilon$ is the order type of G.*

Definition

For any a, b in an ordered group G, they are Achimedean equivalent if there exists $m, n \in \mathbb{N}$ such that $|a^n| \geq |b|$ and $|b^m| \geq |a|$. Denoted, $a \approx b$. Also, a is Archimedean less than b if no such n exists. Denoted $a \ll b$.

Given an ω-copy of G,
$\mathrm{Arch}(G) := \{g \in G : (\forall h \in G)[h <_{\mathbb{N}} g \to \neg(h \approx g)]\}$, and
$W(\mathrm{Arch}(G))$ is the well-ordered initial segment of $\mathrm{Arch}(G)$.

Order Type of Countable Ordered Group

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2. *Let* G *be a countable ordered group. There is a well-order* α *and* $\varepsilon \in \{0,1\}$ *such that* $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$ *is the order type of* G.
3. *Let* G *be a countable abelian ordered group. There is a well-order* α *and* $\varepsilon \in \{0,1\}$ *such that* $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$ *is the order type of* G.

Proof.

Idea: "$\Rightarrow$" Find $\mathrm{Arch}(G)$ and $W(\mathrm{Arch}(G))$. Find the least strictly positive element if possible. Take the quotient and repeat until there is no strictly positive element. □

Order Type of Countable Ordered Group

Proof.

Idea: "$\Leftarrow$" Given a sequence $\{T_i\}_{i\in\mathbb{N}}$ of trees in Baire space, let $T = \{\varepsilon_0\} \cup \{i^\frown\sigma : \sigma \in T_i\}$. Let G be the free abelian group on generators g_σ, $\sigma \in T$. Consider the Kleene-Brouwer order $\mathrm{KB}(T)$ of T. Order the generators so that $g_\sigma \ll g_\tau$ when $\sigma <_{\mathrm{KB}} \tau$. We can show the following:

- The order type of G is $\mathbb{Z}^{\mathrm{KB}(T)}$.
- $\varepsilon = 0$ if and only if $\mathrm{KB}(T)$ is a well-order.
- When $\varepsilon = 1$, T_i has a path if and only if the $\mathbb{Q}$-coordinates of $f(g_{i-1})$ and $f(g_i)$ ($f(e)$ and $f(g_0)$ when $i = 0$) are different.

□

Theorems as problems

Statements like the ones in the previous theorem can be written as follows:

$$(\forall x \in X)(\exists y \in Y)[\varphi(x) \to \psi(x, y)].$$

We can naturally translate it to a computational problem, i.e. given an input x such that $\varphi(x)$, the black box produce an output y such that $\psi(x, y)$.

Notice that for many statements, there could be multiple natural ways to phrase them as a computational problem.

For our purposes, we consider problems on Baire space $\mathbb{N}^{\mathbb{N}}$, i.e. relations $f \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, or equivalently partial multi-valued functions $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$.

Computability on $\mathbb{N}^\mathbb{N}$

Definition

A single-valued function $f :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is computable if there is a total computable function $g : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that:

- $g(\sigma) \preccurlyeq g(\tau)$ when $\sigma \preccurlyeq \tau$,
- $f(x) = y$ if and only if for any n, there exists m such that $y \upharpoonright n \preccurlyeq g(x \upharpoonright m)$.

Definition

A single-valued function $f :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a realizer for a multi-valued function $g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ if

$$(\forall p \in \mathrm{dom}(g))[f(p) \in g(p)].$$

g is computable if it has a computable realizer.

Weihrauch reducibility

Definition

Let f, g be partial multi-valued functions on Baire space. f is Weihrauch reducible to g, denoted $f \leq_{\mathrm{W}} g$ if there are computable Φ, Ψ on Baire space such that:

- given $p \in \mathrm{dom}(f)$, $\Phi(p) \in \mathrm{dom}(g)$, and
- given $q \in g(\Phi(p))$, $\Psi(p, q) \in f(p)$.

Φ, Ψ are called forward functional and backward functional respectively.

$$
\begin{array}{ccc}
p & \xrightarrow{\Phi} & \Phi(p) \\
\Big\downarrow f & & \Big\downarrow g \\
f(p) & \xleftarrow{\Psi(p,\cdot)} & q
\end{array}
$$

Represented space

Weihrauch reducibility is defined in a more general way.

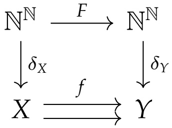
Definition

A represented space is a pair (X, δ_X) where δ_X is a surjection: $\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

If $\delta_X(p) = x$, then we call p a name for x.

A single-valued function F on Baire space is a realizer of a multi-valued function $f :\subseteq X \rightrightarrows Y$ if and only if

$$(\forall p \in \mathrm{dom}(f \circ \delta_X))[\delta_Y \circ F(p) \in f(p)].$$



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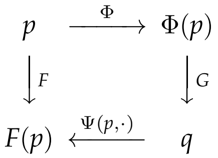
Weihrauch reducibility $f \leq_W g$ is defined using names and realizers:



Algebraic operations

Definition

Let $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Z \rightrightarrows W$, and $h :\subseteq Y \rightrightarrows Z$ be multi-valued functions. We define the following operations:

1. composition $h \circ f :\subseteq X \rightrightarrows Z$,
   $(h \circ f)(x) := \{z \in Z : (\exists y \in f(x))[z \in h(y)]\}$, and
   $\mathrm{dom}(h \circ f) := \{x \in \mathrm{dom}(f) : f(x) \subseteq \mathrm{dom}(h)\}$;
2. product $f \times g :\subseteq X \times Z \rightrightarrows Y \times W$,
   $(f \times g)(x, z) := f(x) \times g(z)$, and
   $\mathrm{dom}(f \times g) := \mathrm{dom}(f) \times \mathrm{dom}(g)$;
3. finite parallelization $f^* :\subseteq X^* \rightrightarrows Y^*$,
   $f^*(i, x) := i \times f^i(x)$, and
   $\mathrm{dom}(f^*) := \mathrm{dom}(f)^*$;
4. parallelization $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$,
   $\widehat{f}(\bigtimes_i x_i) := \bigtimes_{i \in \mathbb{N}} f(x_i)$, and
   $\mathrm{dom}(\widehat{f}) := \mathrm{dom}(f)^{\mathbb{N}}$.

Algebraic operations

Definition

Let $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Z \rightrightarrows W$, and $h :\subseteq Y \rightrightarrows Z$ be multi-valued functions. We define the following operations:

1. parallelization $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$,
   $\widehat{f}(\bigtimes_i x_i) := \bigtimes_{i\in\mathbb{N}} f(x_i)$, and
   $\mathrm{dom}(\widehat{f}) := \mathrm{dom}(f)^{\mathbb{N}}$.

Theorem

Let f and g be problems. The Weihrauch degree
$f * g := \max_{\leq_W}\{f_0 \circ g_0 : f_0 \leq_W f, g_0 \leq_W g\}$ *exists.*

Big five and Weihrauch reducibility

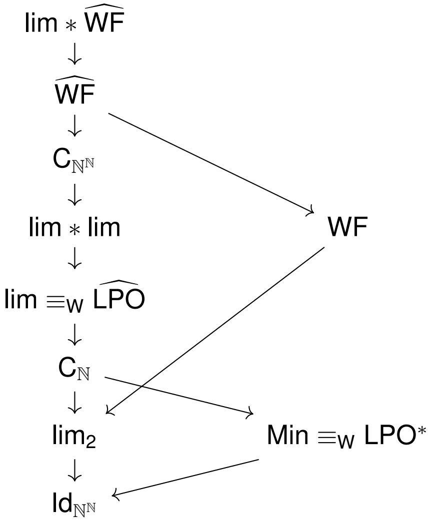
1. RCA_0: $\mathsf{Id}_{\mathbb{N}^\mathbb{N}}$.
2. WKL_0: $\mathsf{C}_{2^\mathbb{N}}$.
3. ACA_0: iterations of lim.
4. ATR_0: many candidates, $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, etc.
5. Π^1_1-CA_0: $\widehat{\mathsf{WF}}$.

Definition

$\mathsf{C}_{\mathbb{N}^\mathbb{N}}$: Given an ill-founded tree in Baire space, find a path through it.

WF: Given a tree in Baire space, tell whether it is well-founded.

A small part of the zoo



Weihrauch problem

1. $\mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f}$ and $\mathsf{AOG} \mapsto \alpha\varepsilon\mathsf{f}$: given a countable (abelian) ordered group G, output the ordinal α and $\varepsilon \in \{0,1\}$ in its order type $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$ with an order-preserving function from G to $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$.
2. $\mathsf{OG} \mapsto \alpha\varepsilon$
3. $\mathsf{OG} \mapsto \varepsilon$
4. $\mathsf{OG} \mapsto \alpha$
5. $\mathsf{OG}\alpha\varepsilon \mapsto \mathsf{f}$: given a countable ordered group G with the ordinal α and $\varepsilon \in \{0,1\}$ in its order type $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$, output an order-preserving function from G to $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$.
6. $\mathsf{OG} \mapsto \alpha 0$ and $\mathsf{OG} \mapsto \alpha 1$: given a countable ordered group G with $\varepsilon \in \{0,1\}$ in its order type $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$, output the ordinal α.

Output everthing

Proposition

$\mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f} \geq_{\mathsf{W}} \widehat{\mathsf{WF}}$

Proof.

□

What about the other direction? Suppose we are given a computable ω-copy of the group, the output we need includes an ω-copy of the ordinal α. Notice that $\widehat{\mathsf{WF}}$ can be used to compute sets that are Π^1_1-complete. Therefore, it is natural to ask the question: what can the computational complexity of α be?

What can α be?

Theorem

If a computable ordered group has order type $\mathbb{Z}^\alpha$, then α is computable.

Proof.

Idea: Build a tree $T \in \mathbb{N}^{<\mathbb{N}}$ by trying to embed $\mathbb{Q}_2 \cap [0,1]$ into the group. This tree has rank $\omega\alpha$. □

Lemma

Given two trees $T_0, T_1 \subseteq \omega^{<\omega}$, if there is a map f from T_1 to T_0 such that $f^{-1}(\sigma)$ has a finite rank as a partial order for any σ, $\mathrm{rk}(f^{-1}(\sigma)) \leq c_l$ for all σ of length l for some constant c_l, and $f(\sigma) \preccurlyeq f(\tau)$ when $\sigma \preccurlyeq \tau$, then $\mathrm{rk}(T_1) \leq^+ \mathrm{rk}(T_0)$. In particular, $\mathrm{rk}(T_1) \leq \mathrm{rk}(T_0)$ when the latter is a limit.

What can α be?

Theorem

If a computable ordered group has order type $\mathbb{Z}^\alpha\mathbb{Q}$, then $\alpha \leq \omega_1^{\mathrm{CK}}$.

Proof.

For each positive element g in the group, we build a tree that tries to embed $\mathbb{Q}_2 \cap [0, 1]$ into the interval between the identity e and g. Then, $\alpha \leq \omega_1^{\mathrm{CK}}$. Otherwise, there is a g mapped to $(0, \dots, 0, 1, 0, \dots)$ where 1 is at the ω_1^{CK} position. □

Theorem

There exists a computable countable ordered group with order type $\mathbb{Z}^{\omega_1^{\mathrm{CK}}}\mathbb{Q}$.

Proof.

There is a group with order type $\mathbb{Z}^{\mathrm{H}}$ where $\mathrm{H} = \omega_1^{\mathrm{CK}}(1 + \mathbb{Q})$ is the Harrison linear order. □

Another way to see this...

Theorem (Gandy Basis Theorem)

If a non-empty set A of reals is Σ^1_1 and, then A contains a real x such that $\omega_1^x = \omega_1^{\mathrm{CK}}$ and $x <_T \mathcal{O}$.

There is some computable input of $\widehat{\mathsf{WF}}$ such that its output is Π^1_1-complete, which computes $\mathcal{O}$.

Feed this input to $\widehat{\mathsf{WF}}$. we get a computable countable ordered group from the forward functional of $\widehat{\mathsf{WF}} \leq_{\mathsf{W}} \mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f}$. If α cannot be non-computable, then the set of order-preserving bijections from the group to its order type is Π^0_2.

By Gandy Basis Theorem, there is one element of this set that cannot compute Kleene's $\mathcal{O}$. Then, the backward functional cannot compute $\mathcal{O}$ using α, ε, f.

One point information

Proposition

$\mathsf{OG} \mapsto \varepsilon \equiv_{\mathsf{W}} \mathsf{WF}$

Proof.

"$\leq_{\mathsf{W}}$" Build a tree T by trying to embed the rationals into the order. Fix a list of rational numbers $\{q_i\}_{i<\omega}$. Define T as follows: any σ is in T if and only if the map from q_i to $\sigma(i) \in G$ for $i < |\sigma|$ preserves the order. Then, T is well-founded if and only if $\varepsilon = 0$.

"$\geq_{\mathsf{W}}$" ▸▸ □

$\mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f} \leq_{\mathsf{W}} \widehat{\mathsf{WF}}$

Proof.

"$\leq_{\mathsf{W}}$" Assume that the input of $\mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f}$ is computable and $\varepsilon = 1$.

- The forward functional simply makes countably many trees so that the output of $\widehat{\mathsf{WF}}$ is Π^1_1-complete.
- The backward functional will identify which $\mathbb{Z}^\alpha$ copy each group element is in.
- It will also make new guesses of α for a copy of $\mathbb{Z}^\alpha$ whenever a new group element is known to be in this copy.
- It will build the partial order-preserving map according to the current guess of α.
- All the questions the backward functional ask in order to do so are Π^1_1.

□

On the side

Proposition

- $\mathsf{OG} \mapsto \alpha \not\geq_{\mathsf{W}} \mathsf{lim}_2$.
- $\mathsf{OG} \mapsto \alpha\varepsilon \not\geq_{\mathsf{W}} \mathsf{lim}_2 \times \mathsf{lim}_2$.

Definition

$\mathsf{lim}_2 :\subseteq 2^{\mathbb{N}} \to 2$ is the limit operation on $\{0, 1\}$.

First-order part

Definition (Dzhfarov, Solomon, & Yokoyama)

For a Weihrauch problem P, the first-order part of P, denoted by ${}^1\mathsf{P}$, is the following first-order problem:

- the ${}^1\mathsf{P}$-instances are all triples $\langle f, \Phi, \Psi \rangle$, where $f \in \omega^\omega$ and Φ and Ψ are Turing functionals such that $\Phi(f) \in \mathrm{dom}(\mathsf{P})$ and $\Psi^{f \oplus g}(0)\downarrow$ for all $g \in \mathsf{P}(\Phi(f))$;
- the ${}^1P$-solutions to any such $\langle f, \Phi, \Psi \rangle$ are all y such that $\Psi^{f \oplus g}(0) \downarrow = y$ for some $g \in \mathsf{P}(\Phi(f))$.

$$\begin{array}{ccc} \langle f, \Phi, \Psi \rangle & \xrightarrow{\ \Phi\ } & \Phi(f) \\ \Big\downarrow {}^1P & & \Big\downarrow P \\ y = \Psi^{f \oplus g}(0)\downarrow & \xleftarrow{\ \Psi\ } & g \end{array}$$

Intuitively, the first-order part of a problem P is the strongest problem with codomain ω that is Weihrauch reducible to P.

First-order part

Definition

$$\mathsf{LPO} : \mathbb{N}^{\mathbb{N}} \to \{0,1\},\ \mathsf{LPO}(p) = \begin{cases} 0 \text{ if } (\exists k)[p(k) = 0], \\ 1 \text{ otherwise.} \end{cases}$$

Theorem (Brattka, Gherardi, Marcone, & Pauly)

- $\mathsf{LPO}^* \equiv_{\mathrm{W}} \mathsf{Min}$.
- $\lim_2 |_{\mathrm{W}} \mathsf{LPO}^*$.
- $\lim_2, \mathsf{LPO}^* <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.

Proposition

- ${}^1\mathrm{OG} \mapsto \alpha \equiv_{\mathrm{W}} \mathsf{LPO}^*$.
- ${}^1\mathrm{OG} \mapsto \alpha\varepsilon \equiv_{\mathrm{W}} \mathsf{LPO}^* \times \mathsf{WF}$.

Corollary

$\mathrm{OG} \mapsto \alpha \not\geq_{\mathrm{W}} \lim_2$. $\mathrm{OG} \mapsto \alpha\varepsilon \not\geq_{\mathrm{W}} \lim_2 \times \lim_2$.

How much is needed to output α?

Proposition

$\mathsf{OG}\alpha\varepsilon \mapsto \mathsf{f} \leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

Idea: build a tree such that the first level maps the first element of the group to a point in $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$, and the second level maps the first point of $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$ to an element of the group.

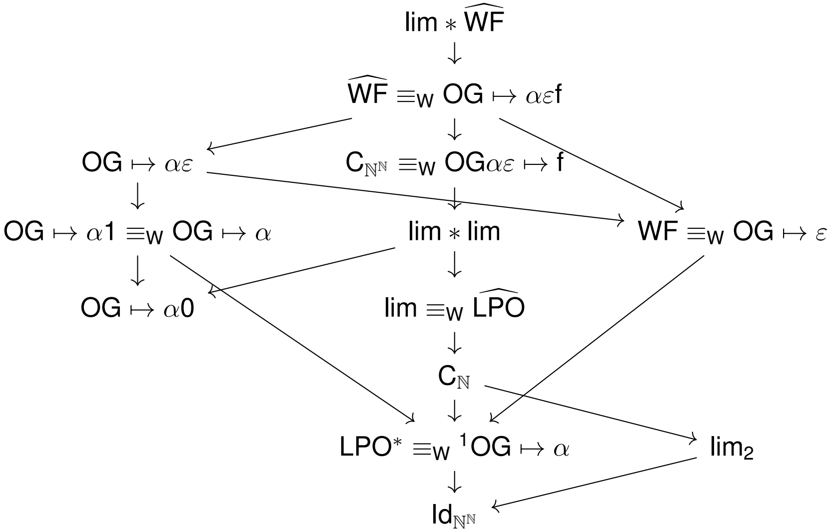
Proposition

$\mathsf{OG} \mapsto \alpha\varepsilon \not\leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} * \mathsf{WF}$

Corollary

$\mathsf{OG} \mapsto \alpha \not\leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$, $\mathsf{OG} \mapsto \alpha 1 \not\leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$, $\mathsf{OG} \mapsto \alpha 0 \leq_{\mathsf{W}} \mathsf{lim} * \mathsf{lim}$.

A slightly bigger part of the zoo



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Thank You!