O-minimal Coherence

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Background

Let $\mathcal{O}_{\mathbb{C}^n}$ denote the sheaf of rings where $\mathcal{O}_{\mathbb{C}^n}(U)$ is the ring of holomorphic functions defined on U, for each $U \subseteq \mathbb{C}^n$ open. It's also an $\mathcal{O}_{\mathbb{C}^n}$-module.
In complex analysis, it is well-known that

Fact

[Oka50] (Oka) For any positive integer n, $\mathcal{O}_{\mathbb{C}^n}$ is a coherent $\mathcal{O}_{\mathbb{C}^n}$-module. i.e. $\mathcal{O}_{\mathbb{C}^n}$ satisfies that

1. *$\mathcal{O}_{\mathbb{C}^n}$ locally finite.*
2. *Every relation sheaf of $\mathcal{O}_{\mathbb{C}^n}$ is locally finite.*

Background

This result is generalized in [PS08] to the case of any algebraically closed field $\mathcal{K}$ of characteristic 0.

Fact

(Peterzil, Starchenko) For any positive integer n, $\mathcal{O}_{\mathcal{K}^n}$ *is a coherent* $\mathcal{O}_{\mathcal{K}^n}$*-module.*

In the above results, a sheaf means the usual sheaf in e.g. [Har13, Chapter II].

Background

In [BBT22], coherence theorem is proved on the site $\underline{\mathbb{C}^n}$ where the coverings are finite coverings by definable open sets:

Fact

(Bakker, Brunebarbe, Tsimerman) The definable structure sheaf $\mathcal{O}_{\mathbb{C}^n}$ of $\mathbb{C}^n$ is a coherent $\mathcal{O}_{\mathbb{C}^n}$-module (as a sheaf on the site $\underline{\mathbb{C}^n}$).

(The sheaves in [BBT22] are different from the usual sheaves defined in [Har13]. We will explain more later.)

We use different techniques to prove the coherence of $\mathcal{O}_{\mathcal{K}^n}$ as a sheaf on the site $\underline{\mathcal{K}^n}$, where $\mathcal{K}$ is an algebraically closed field of characteristic 0.

Theorem

The definable structure sheaf $\mathcal{O}_{\mathcal{K}^n}$ of $\mathcal{K}^n$ is a coherent $\mathcal{O}_{\mathcal{K}^n}$-module as a sheaf on the site $\underline{\mathcal{K}^n}$.

Preliminaries

Setting

(The same setting as in [PS01].) Let $\mathcal{K}$ be an algebraically closed field of characteristic zero. Then $\mathcal{K} = \mathcal{R}(\sqrt{-1})$ for some real closed subfield $\mathcal{R}$. Such $\mathcal{R}$ is not unique. We fix one such $\mathcal{R}$ and fix an o-minimal expansion of the chosen real closed field. The topology on $\mathcal{R}$ is generated by the definable open intervals. The topology on $\mathcal{K}$ is identified with that on $\mathcal{R}^2$. When we say definable, we mean definable in the o-minimal structure $\mathcal{R}$ with parameters in $\mathcal{R}$.

After defining the topological structure on $\mathcal{K}$, we define the differential structure:

Preliminaries

For one variable, differentiability is defined as follows:

Definition

[PS03, Definition 2.1.] Let $U \subseteq \mathcal{K}$ be a definable open set and $F : U \to \mathcal{K}$ a definable function. Let $z_0 \in U$. We say that F is *$\mathcal{K}$-differentiable at z_0* if the limit as z tends to z_0 in $\mathcal{K}$ of $(f(z) - f(z_0))/(z - z_0)$ exists in $\mathcal{K}$ (all operations taken in $\mathcal{K}$, while the limit is taken in the topology induced on $\mathcal{K}$ by $\mathcal{R}^2$).

Preliminaries

For multi-variables, differentiability is defined as follows:

Definition

[PS03, Definition 2.8.] Let $V \subseteq \mathcal{K}^n$ be a definable open set, $F : V \to \mathcal{K}$ a definable map. F is called $\mathcal{K}$-*differentiable* on V if it is continuous on V and for every $(z_1, ..., z_n) \in V$ and $i = 1, ..., n$, the function $F(z_1, ..., z_{i-1}, -, z_{i+1}, ..., z_n)$ is $\mathcal{K}$-differentiable in the i-th variable at z_i (in other words, F is continuous on V and $\mathcal{K}$-differentiable in each variable separately).

Spectral Topology

The spectral topology is a topology on the type space:

Definition

[EJP06, Definition 2.2.] Let $X \subseteq \mathcal{R}^m$ be a definable set (with parameters in $\mathcal{R}$). *The o-minimal spectrum* $\tilde{X}$ *of* X is the set of complete m-types $S_m(\mathcal{R})$ of the first order theory $Th_{\mathcal{R}}(\mathcal{R})$ which imply a formula defining X. This is equipped with the topology generated by the basic open sets of the form $\tilde{U} = \{\alpha \in \tilde{X} : U \in \alpha\}$, where U is a definable, relatively open subset of X, and $U \in \alpha$ means the formula defining U is in α. We call this topology on X the *spectral topology*.

Sheaves on $S_n(\mathcal{K})$

Let $S_n(\mathcal{K})$ denote $S_{2n}(\mathcal{R})$. We use this unconventional notation to emphasize that we are considering functions on $\mathcal{K}^n$.

Definition

For each definable open set $U \subseteq \mathcal{K}^n$, let $\mathcal{O}_{\mathcal{K}^n}(\tilde{U})$ denote the ring of $\mathcal{K}$-differentiable functions defined on U. It's easy to check that this defines a sheaf on $S_n(\mathcal{K})$.

Let $\mathcal{O}_{\mathcal{K}^n}$ denote the sheaf of rings where $\mathcal{O}_{\mathcal{K}^n}(\tilde{U})$ is the ring of $\mathcal{K}$-differentiable functions defined on U, for each $U \subseteq \mathcal{K}^n$ open.

Given $p \in S_n(\mathcal{K})$, let $\mathcal{O}_p$ denote the set of germs for functions

$$\{f : U \to \mathcal{K} : U \text{ is some open definable set such that } p \in \tilde{U} \text{ and } f \text{ is } \mathcal{K}\text{-holomophic on } U\}.$$

Sheaves on $S_n(\mathcal{K})$

Definition

Given a definable set $A \subseteq \mathcal{K}^n$, let $\mathcal{I}_p(A) \subseteq \mathcal{O}_p$ denote the set of germs for functions

$$\{f : U \to \mathcal{K} : U \text{ is some open definable set such that } p \in \tilde{U},$$

$$f \text{ is } \mathcal{K}\text{-holomophic on } U \text{ and } \forall x \in A \cap U,\ f(x) = 0\ \}.$$

Let $g_1, ..., g_t \in \mathcal{O}_p$. Let $R_p(g_1, ..., g_t)$ denote the set

$$\{(f_1, ..., f_t) \in \mathcal{O}_p^t : f_1 g_1 + ... + f_t g_t = 0\}.$$

O-minimal Site

We translate definitions about sites in [Stacks] into o-minimal context:

Definition

[Stacks, Part 1, Chapter 7, Definition 6.2] Let $X \subseteq \mathcal{K}^n$ be a definable set. The *o-minimal site* $\underline{X}$ on X consists of definable (relative) open subsets of X, together with $Cov(X) := \{(U, \{U_i\}_{i=1}^k) : U, U_1, ..., U_k \subseteq X$ definable (relative) open, $\{U_i\}_{i=1}^k$ a finite covering of $U\ \}$.

Presheaf

Definition

[Stacks, Part 1, Chapter 6, Section 5] A *presheaf* of abelian groups (resp. rings) on an o-minimal site $\underline{X}$ is defined the same as usual: Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups (resp. rings) on an o-minimal site $\underline{X}$ consists of the following data:

(a) a collection of non empty abelian groups (resp. rings) $\mathcal{F}(U)$ associated with every definable open set $U \subseteq X$,

(b) a collection of morphisms of abelian groups (resp. rings) $\rho_{U,V} : \mathcal{F}(V) \to \mathcal{F}(U)$ defined whenever $U \subseteq V$ and satisfying the transitivity property, $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$ for $U \subseteq V \subseteq W$, $\rho_{U,U} = Id_U$ for every U.

Presheaf

Definition

[Stacks, Part 1, Chapter 6, Definition 6.1] Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on the o-minimal site $\underline{X}$. A *presheaf of $\mathcal{O}$-modules* $\mathcal{F}$ on an o-minimal site $\underline{X}$ is a presheaf $\mathcal{F}$ of abelian groups with the following additional data:

(a) For every definable open set $U \subseteq X$, $\mathcal{F}(U)$ is a non empty $\mathcal{O}(U)$-module;

(b) for every definable open $U \subseteq X$ the $\mathcal{O}(U)$-module structure of $\mathcal{F}(U)$ is compatible with restriction mappings (of $\mathcal{F}$ and $\mathcal{O}$). i.e. for definable open $U \subseteq V \subseteq X$, $r \in \mathcal{O}(V)$, $x \in \mathcal{F}(V)$, $\rho_{U,V}(r)\tau_{U,V}(x) = \tau_{U,V}(rx)$, where ρ, τ are the restriction mappings of $\mathcal{F}$ and $\mathcal{O}$ resp.

Sheaf

Definition

[Stacks, Part 1, Chapter 6, Definition 7.1.] Let $\underline{X}$ be an o-minimal site, and let $\mathcal{F}$ be a presheaf of abelian groups (resp. rings, $\mathcal{O}$-modules) on $\underline{X}$. We say $\mathcal{F}$ is a *sheaf* if for every definable open $U \subseteq X$ and every definable open finite covering $\{U_i\}_{i=1}^k$ of U,

(i) if $(s_i)_{i=1}^k$ satisfies $s_i \in \mathcal{F}(U_i)$ for each i and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for each pair i, j, then there is a unique $s \in U$ such that $s|_{U_i} = s_i$ for each i;

(ii) for $s, t \in \mathcal{F}(U)$, if $s|_{U_i} = t|_{U_i}$ for each i then $s = t$.

Morphism

Definition

[Stacks, Part 1, Chapter 7. Definition 11.1.] Let $\underline{X}$ be an o-minimal site, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves of modules (i.e. compatible with restiction mappings).

(1) We say that φ is *injective* if for every definable open $U \subseteq X$ the map $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.

(2) We say that φ is *surjective* if for every definable open $U \subseteq X$ and every section $s \in \mathcal{G}(U)$ there exists a finite covering $\{U_i\}_{i=1}^{k}$ of U such that for each i, U_i is definable open and the restriction $s|_{U_i}$ is in the image of $\varphi : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

Coherence

Definition

([BBT22, Definition 2.13]) Let $\underline{X}$ be an o-minimal site. Given an $\mathcal{O}_X$-module M, we say that M is of *finite type* (as an $\mathcal{O}_X$-module) if there exists a finite definable open (relative to X) cover X_i of X and surjections $\mathcal{O}_{X_i}^n \twoheadrightarrow M_{X_i}$ for some positive integer n on each of those open sets. We say that M is *coherent* (as an $\mathcal{O}_X$-module) if it is of finite type, and given any definable open $U \subseteq X$ and any $\mathcal{O}_U$ -module homomorphism $\varphi : \mathcal{O}_U^n \to M_U$, the kernel of φ is of finite type.

Coherence

Remark

By definitions above, given a definable open U and an $\mathcal{O}_U$-module M, to show that M is of finite type, it suffices to show that there exist a finite family of definable open sets $U_1, ..., U_k$ covering U and sheaf morphisms $\varphi_i : \mathcal{O}_{U_i}^{N_i} \to M_{U_i}$, $i = 1, ..., k$ satisfying that for any fixed i, for any definable open $V \subseteq U_i$ and every section $s \in M(V)$, there exist a finite family of definable open sets $V_1, ..., V_l$ covering V and for each $j \in \{1, ..., l\}$, $t_j \in \mathcal{O}_{U_i}(V_j)$ such that $\varphi(V_j)(t_j) = s|_{V_j}$.

Motivation

Definition

[EJP06, Definition 3.1.] We denote by $Sh_{dtop}(X)$ the category of sheaves of abelian groups on the o-minimal site $\underline{X}$ and by $Sh(\tilde{X})$ the category of usual sheaves of abelian groups on $\tilde{X}$.

The following fact is the motivation for our proof. It says a sheaf on the site $\underline{X}$ is the same as a usual sheaf in [Har13, Chapter II] on the space $\tilde{X} \subseteq S_n(\mathcal{K})$ with spectral topology.

Fact

[EJP06, Proposition 3.2] $Sh(\tilde{X})$ and $Sh_{dtop}(X)$ are isomorphic.

Number of Zeroes

Let $\pi : \mathcal{K}^n \to \mathcal{K}^{n-1}$, $\pi_n : \mathcal{K}^n \to \mathcal{K}$ denote the projection onto the first $(n-1)$ coordinates and the projection onto the n-th coordinate resp.

Lemma

Let $p \in S_n(\mathcal{K})$. Fix $f \in \mathcal{O}_p$ and an open neighborhood U of p on which f is defined and is $\mathcal{K}$-differentiable. Suppose for all $y \in \pi(U)$, there are finitely many zeroes of $f(y,-)$ in $U_y := \{x \in \mathcal{K} : (y,x) \in U\}$, counting multiplicity.
Then there exist $i \in \mathbb{N}$ and $V \subseteq U$ a definable open neighborhood of p such that for any $y \in \pi(V)$, there are exactly i zeroes of $f(y,-)$ in V_y counting multiplicity.

Number of Zeroes

Fact

[PS01, Theorem 2.56.] Let $W \subseteq \mathcal{R}^n$, $U \subseteq \mathcal{K}$ be definable open sets, $F : U \times W \to \mathcal{K}$ a definable continuous function such that for every $w \in W$, $F(-, w)$ is a $\mathcal{K}$- differentiable function on U. Take $(z_0, w_0) \in U \times W$ and suppose that z_0 is a zero of order m of $F(-, w_0)$.

Then for every definable neighborhood V of z_0 there are definable open neighborhoods $U_1 \subseteq V$ of z_0 and $W_1 \subseteq W$ of w_0 such that $F(-, w)$ has exactly m zeroes in U_1 (counted with multiplicity) for every $w \in W_1$.

Number of Zeroes

Fact

[Cos00, Theorem 6.11] (Definable Tubular Neighborhood) Let M be a definable C^k submanifold of $\mathcal{R}^n$. There exists a definable open neighborhood U of the zero-section $M \times \{0\}$ in the normal bundle NM such that the restriction $\varphi|U$ is a C^{k-1} diffeomorphism onto an open neighborhood Ω of M in $\mathcal{R}^n$. Moreover, we can take U of the form

$$U = \{(x, v) \in NM : \|v\| < \epsilon(x)\},$$

where ϵ is a positive definable C^k function on M.

Coherence on Type Space

Theorem

(type version of [PS08, Theorem 11.3.]) Assume that A is a $\mathcal{K}$-analytic subset of $U \subseteq \mathcal{K}^n$ and assume that $G_1, \ldots, G_t$ are $\mathcal{K}$-holomorphic maps from A into $\mathcal{K}^N$. Then we can write A as a union of finitely many relatively open sets $A_1, \ldots, A_m$ such that on each A_i the following holds:
There are finitely many tuples of $\mathcal{K}$-holomorphic functions on A_i,

$$\{(H_{j,1}, \ldots, H_{j,t}) : j = 1, \ldots, k\}, k = k(i),$$

with the property that for every $p \in S_n(\mathcal{K})$ with $p \in \tilde{A}_i$, the module $R_p(g_1, \ldots, g_t)$ equals its submodule generated by $\{(h_{j,1}, \ldots, h_{j,t}) : j = 1, \ldots, k\}$ over $\mathcal{O}_p$ (where g_i and $h_{i,j}$ are the germs of G_i and $H_{i,j}$ at p, resp).

Main Theorem

Theorem

(o-minimal version of [BBT22, Theorem 2.21]) The definable structure sheaf $\mathcal{O}_{\mathcal{K}^n}$ of $\mathcal{K}^n$ is a coherent $\mathcal{O}_{\mathcal{K}^n}$-module as a sheaf on the site $\underline{\mathcal{K}^n}$.

Proof.

Use compactness and type version of [PS08, Theorem 11.3], to check the definition. □

Remark

Let $X \subseteq \mathcal{K}^n$ be definable open. Let $Sh^{\mathcal{O}(X)}(\tilde{X})$ denote the category of sheaves of $\mathcal{O}(X)$-modules on $\tilde{X}$. Let $Sh^{\mathcal{O}(X)}_{dtop}(X)$ denote the category of sheaves of $\mathcal{O}(X)$-modules on $\underline{X}$ as an o-minimal site. We show that $Sh^{\mathcal{O}(X)}(\tilde{X})$ and $Sh^{\mathcal{O}(X)}_{dtop}(X)$ are isomorphic categories, and the surjective maps are exactly the epimorphisms in both categories. Hence, from a category-theoretic perspective, type version of [PS08, Theorem 11.3.] immediately implies the main theorem.

Remark

Lemma

[Stacks, Part 1, Chapter 7, Lemma 11.2.] The surjective maps defined above are exactly the epimorphisms of the category $Sh_{dtop}^{\mathcal{O}(X)}(X)$.

Lemma

The surjective maps (i.e. surjective at the stalks) are exactly the epimorphisms of the category $Sh^{\mathcal{O}(X)}(\tilde{X})$.

Proposition

$Sh^{\mathcal{O}(X)}(\tilde{X})$ and $Sh_{dtop}^{\mathcal{O}(X)}(X)$ are isomorphic categories.

Remark

Another proof of the main theorem:
Let $\iota : Sh^{\mathcal{O}(X)}(\tilde{X}) \to Sh^{\mathcal{O}(X)}_{dtop}(X)$ be an isomorphism. Let $U \subseteq \mathcal{K}^n$ be definable open and $\varphi : \mathcal{O}_U^m \to \mathcal{O}_U$ a $\mathcal{O}_U$-module homomorphism. By the type version of [PS08, Theorem 11.3.], there exists a finite definable open covering $\{U_i\}_{i=1}^k$ of U such that for some $l \in \mathbb{N}$ and for each $i \in \{1, ..., k\}$, there exists $\psi_i : \mathcal{O}^l_{\tilde{U}_i} \twoheadrightarrow ker(\iota^{-1}(\varphi))_{\tilde{U}_i}$. Since surjective morphisms are epimorphisms in $Sh^{\mathcal{O}(X)}(\tilde{X})$, $\iota(\psi_i) : \mathcal{O}^l_{U_i} \to ker(\varphi)_{U_i}$ is an epimorphism and hence a surjective morphism.

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