Bounds on the dimension of lineal extensions

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Outline

1. Geometric measure theory: the classical picture
2. Geometric measure theory: the effective picture
3. The line segment extension problem and proof of the main theorem
4. Related results

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Problems in Geometric Measure Theory

The basic question: How large are sets?

- Usually, by size we mean measure or some notion of dimension (e.g. Hausdorff, packing, box-counting, Assouad...).

- More precisely, what can we say about sets that we know have a certain geometric property, or that are obtained by some natural geometric operation on a set with known size?

Covers and packings

Let $E \subset \mathbb{R}^n$ and $\{B_i\}_{i\in\mathbb{N}}$ be a collection of open balls in $\mathbb{R}^n$.

We call $\{B_i\}_{i\in\mathbb{N}}$ a δ-cover for E if

- $E \subseteq \bigcup_{i=1}^{\infty} B_i$
- $\operatorname{diam}(B_i) \leq \delta$

We call $\{B_i\}_{i\in\mathbb{N}}$ a δ-packing for E if

- The balls are pairwise disjoint
- The balls have centers in E
- $\operatorname{diam}(B_i) \leq \delta$

Hausdorff and packing dimension

Hausdorff dimension

$$\mathcal{H}^s_\delta(E) = \inf_{\delta-\text{covers}} \{\sum_{i=1}^{\infty} \text{diam}(B_i)^s\}$$

$$\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E)$$

$$\dim_H(E) = \inf\{s : \mathcal{H}^s(E) = 0\}$$

Packing dimension

$$\bar{\mathcal{P}}^s_\delta(E) = \sup_{\delta-\text{packings}} \{\sum_{i=1}^{\infty} \text{diam}(B_i)^s\}$$

$$\bar{\mathcal{P}}^s(E) = \lim_{\delta \to 0^+} \bar{\mathcal{P}}^s_\delta(E)$$

$$\mathcal{P}^s(E) = \inf\{\sum_{i=1}^{\infty} \bar{\mathcal{P}}^s(E_i) : E \subseteq \bigcup_{i=1}^{\infty} E_i\}$$

$$\dim_P(E) = \inf\{s : \mathcal{P}^s(E) = 0\}$$

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Complexity of points in Euclidean space

Fix a universal prefix-free oracle Turing machine U. Given $A \subseteq \mathbb{N}$, the (prefix-free) Kolmogorov complexity of a string σ relative to A is

$$K^A(\sigma) = \min\{|\pi| : U^A(\pi) = \sigma\}$$

We can encode rational vectors $q \in \mathbb{R}^n$ as binary strings, and hence can talk about $K^A(q)$. This in turn allows us to define the complexity of *arbitrary* points in $\mathbb{R}^n$ at any given precision.

$$K_r^A(x) = \min\{K^A(q) : q \in B_{2^{-r}}(x)\}$$

Effective dimension

Definition

The effective Hausdorff dimension of a point $x \in \mathbb{R}^n$ relative to an oracle $A \subseteq \mathbb{N}$ is given by

$$\dim^A(x) = \liminf_{r \to \infty} \frac{K_r^A(x)}{r}$$

Definition

The effective packing dimension of a point $x \in \mathbb{R}^n$ relative to an oracle $A \subseteq \mathbb{N}$ is given by

$$\mathrm{Dim}^A(x) = \limsup_{r \to \infty} \frac{K_r^A(x)}{r}$$

The point-to-set principle

Effective dimension is directly related to classical dimension through the following "point-to-set" principle(s):

Theorem (J. Lutz and N. Lutz, 2015)

For all $E \subset \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x)$$

and

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x)$$

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The Kayeya conjecture

Call $E \subseteq \mathbb{R}^n$ a **Besicovitch set** if E contains a unit line segment in every direction.

This is a geometric requirement that should force sets to be large, but how large?

Kakeya conjecture

Any Besicovitch set $E \subseteq \mathbb{R}^n$ has Hausdorff dimension n.

Question: Does it matter if we use line segments or full lines in the definition of Besicovitch sets?

Line segment extension

More general question: Does the dimension of a set of line segments increase when we replace them with full lines?

- In 2014, Keleti showed (in the plane) that Hausdorff dimension does not increase under line segment extension. He also showed this is false for box dimension.

Theorem (Bushling and F., 2024)

If $E \subseteq \mathbb{R}^2$ is a union of Hausdorff dimension 1 subsets of lines, and $L(E)$ is the union of the corresponding full lines, then $\dim_P(E) = \dim_P(L(E))$

Effectivization of line segment extension

- Using the point-to-set principle

$$\dim_P(E) = \min_{A\subseteq\mathbb{N}} \sup_{z\in E} \mathrm{Dim}^A(z) = \min_{A\subseteq\mathbb{N}} \sup_{\ell\in\Lambda} \sup_{z\in E_\ell} \mathrm{Dim}^A(z)$$

$$\dim_P(L(E)) = \min_{A\subseteq\mathbb{N}} \sup_{z\in L(E)} \mathrm{Dim}^A(z) = \min_{A\subseteq\mathbb{N}} \sup_{\ell\in\Lambda} \sup_{z\in\ell} \mathrm{Dim}^A(z).$$

- So it suffices to show that for every $A \subseteq \mathbb{N}$, $a, b \in \mathbb{R}$, and S a Hausdorff dimension 1 subset of $\mathbb{R}$,

$$\sup_{x\in S} \mathrm{Dim}^A(x, ax+b) = \sup_{x\in\mathbb{R}} \mathrm{Dim}^A(x, ax+b)$$

More effectivization

- The points in S that we want to use are the ones that are complex relative to (A, a, b).

- By the point-to-set principle, for any $\varepsilon > 0$, we can find a point $x_\varepsilon \in S$ satisfying
$$\dim^{A,a,b}(x_\varepsilon) \geq 1 - \varepsilon$$

- Hence, it suffices to show that
$$\lim_{\varepsilon \to 0} \mathrm{Dim}^A(x_\varepsilon, ax_\varepsilon + b) = \sup_{x \in \mathbb{R}} \mathrm{Dim}^A(x, ax + b)$$

Ideas of the pointwise proof

- The reason we needed the **Hausdorff** dimension of x_ε to be large is that this guarantees that at every sufficiently large precision

$$K_r^{A,a,b}(x_\varepsilon) \geq r - 2\varepsilon r$$

  i.e. close to maximal. This lets us focus on the complexity properties of the line (a, b).

- Most of the proof entails understanding how the complexity function $K_r^A(a, b)$ relates to upper bounds on $K_r^A(x, ax + b)$ and lower bounds on $K_r^A(x_\varepsilon, ax_\varepsilon + b)$.

Partitioning to get upper bounds

Suppose $t < r$. Then by symmetry of information,

$$K_r^A(x, ax+b) \approx K_t^A(x, ax+b) + K_{r,t}^A(x, ax+b)$$

So we can consider the complexity of (a, b) on *intervals* of precisions.

On the first interval: $K_t^A(x, ax+b) \lessapprox K_t^A(a, b, x) \lessapprox K_t^A(a, b) + t$

On arbitrary intervals: $K_{r,t}^A(x, ax+b) \lessapprox 2(r-t)$

Let c_r be a minimizer of $K_t^A(a, b) - t$. Then combining these bounds,

$$K_r^A(x, ax+b) \lessapprox K_{c_r}^A(a, b) + c_r + 2(r - c_r)$$

Lower bounds

We want to show that this upper bound is essentially a lower bound for the points $(x_\varepsilon, ax_\varepsilon + b)$ as well, which will complete the proof. This is harder, and relies on two remarkable lemmas of N. Lutz and Stull.

- **First lemma:** An "enumeration" lemma that gives sufficient technical conditions for $K_r^A(x, ax + b) \gtrapprox K_r^A(x, a, b)$. The main condition is that for lines (u, v) such that $ux + v = ax + b$, (u, v) is either close to (a, b) or has appreciably higher complexity
- **Second lemma:** A geometric lemma. If $ax + b = ux + v$, and (a, b) agrees with (u, v) to precision s,

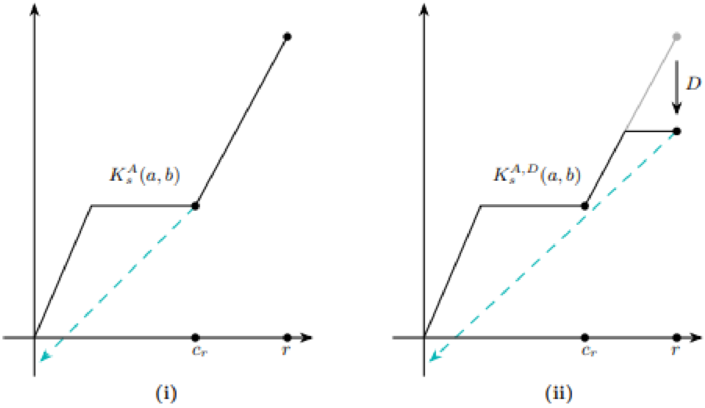
$$K_r^A(u, v) \geq K_s^A(a, b) + K_{r-s,r}^A(x|a, b)$$

The second lemma will allow us to show the key condition of the first lemma is satisfied, since we picked x_ε such that

$$K_r^A(u, v) \geq K_s^A(a, b) + K_{r-s,r}^A(x_\varepsilon|a, b) \geq K_s^A(a, b) + (1 - 2\varepsilon)(r - s)$$

Choosing an appropriate oracle

We may need to reduce the complexity of (a, b) so that the conditions of the enumeration lemma hold. We can temporarily introduce an oracle D for this purpose.



Finishing the proof of the main theorem

This teal property ensures $K^{A,D}_s(a,b)$ is appropriately large compared to $K^{A,D}_r(a,b)$, hence

$$K^{A,D}_r(u,v) \geq K^{A,D}_s(a,b) + (1-2\varepsilon)(r-s)$$

is large enough that we can apply the enumeration lemma, which gives the bound

$$\begin{aligned} K^A_r(x_\varepsilon, ax_\varepsilon + b) &\geq K^{A,D}_r(x_\varepsilon, ax_\varepsilon + b) \\ &\gtrapprox K^{A,D}(a,b,x_\varepsilon) \\ &\gtrapprox K^{A,D}_r(a,b) + r \\ &\geq K^A_{c_r}(a,b) + c_r + 2(r - c_r) \end{aligned}$$

Comparing to the *upper* bound for *arbitrary* x completes the proof.

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The generalized Kakeya conjecture for packing dimension in the plane

Theorem (Bushling and F., 2024)

Let $E \subseteq \mathbb{R}^2$ and let $D \subseteq S^1$ be the set of directions of lines intersecting E in a set of Hausdorff dimension 1. If $D \neq \varnothing$, then

$$\dim_P D + 1 \leq \dim_P E.$$

Using a similar reduction, it suffices to show that for all $a, b \in \mathbb{R}$ and $A \subseteq \mathbb{N}$,

$$\mathrm{Dim}^A(a) + 1 \leq \lim_{\varepsilon \to 0} \mathrm{Dim}^A(x_\varepsilon, ax_\varepsilon + b)$$

Using our lower bound,

$$\begin{aligned} K_r^A(x_\varepsilon, ax_\varepsilon + b) &\gtrapprox K_{c_r}^A(a, b) + c_r + 2(r - c_r) \\ &\geq K_{c_r}^A(a) + c_r + 2(r - c_r) \\ &\geq K_r^A(a) + r \end{aligned}$$

Bounds in higher dimensions

Theorem (Bushling and F., 2024)

Let $E \subseteq \mathbb{R}^n$ be a union of line segments, and $L(E)$ be the union of the corresponding full lines. Then

$$\dim_P(L(E)) \leq 2\dim_P(E) - 1$$

Idea: if $z = x + t(y - x)$, then $K_r^A(z) \lessapprox K_r^A(x, y, t)$. Automatically, this gives

$$\dim_P(L(E)) \leq 2\dim_P(E) + 1$$

But, we can freely choose the first coordinates of x and y so that x, y, and t share lots of information.

Sketch of the proof

First, choose y so that its first coordinate is the same as z's after some precision. t is easily computed from the first coordinates of x, y, and z, so

$$K_r^A(x, y, t) \approx K_r^A(x, y)$$

Lemma

For all $y \in \mathbb{R}^n$, $A \subseteq \mathbb{N}$, and $\varepsilon > 0$, there exists a dense set of points $x \in \mathbb{R}$ such that, for all sufficiently large r (depending on x),

$$K_r^A(y|x) \leq \max\left\{K_r^A(y) - (1-\varepsilon)r, \varepsilon r\right\}. \tag{1}$$

Choose x so that its first coordinate helps in the computation of y as above, which will ensure

$$K_r^A(x, y) \approx K_r^A(x) + K_r^A(y) - r \tag{2}$$

Further directions

Keleti, 2014

If the line segment extension conjecture (for Hausdorff dimension) holds, then Besicovitch sets have Hausdorff dimension at least $n - 1$ and packing dimension n.

The proof of the above is quantitative, so bounds on how much the Hausdorff dimension increases under line segment extension imply bounds on the Kakeya conjecture.

Thank you!