Continuous Combinatorics of Countable Abelian Group Actions

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Graph combinatorics

A graph is a pair $G = (V, E)$, where V is a set (vertices) and E is a set of unordered pairs of elements of V (edges).

The chromatic number of a graph G, denoted $\chi(G)$, is the smallest cardinality of a set C (colors) such that there exists a (proper coloring) map $c\colon V(G) \to C$ with $c(x) \neq c(y)$ if $xy \in E(G)$.

The edge chromatic number of G, denoted $\chi'(G)$, is the smallest cardinality of a set C such that there exists a (proper edge coloring) map $c\colon E(G) \to C$ with $c(e) \neq c(f)$ if $e \neq f$ and $e \cap f \neq \varnothing$.

A perfect matching of G is a set $M \subseteq E(G)$ (viewed as an induced subgraph of G) such that $V(M) = V(G)$ and every vertex has degree 1 in M.

Topological graph combinatorics

A topological graph is a graph G where $V(G)$ is a topological space.

The continuous chromatic number of a topological graph G, denoted $\chi_c(G)$, is the smallest cardinality of a set C such that there exists a continuous proper coloring $c\colon V(G) \to C$, where C has the discrete topology.

The continuous edge chromatic number of a topological graph G, denoted $\chi'_c(G)$, is the smallest cardinality of a set C such that there exists a continuous proper edge coloring $c\colon E(G) \to C$, where C has the discrete topology.

A perfect matching M of G is clopen (open, etc.) if M is a relatively clopen (open, etc.) subset of $E(G)$.

An example: irrational rotation

Let $\alpha \in (0,1)$ be an irrational number. Define a graph $G = (\mathbb{T}, E)$, where

$$xy \in E \iff y/x = e^{\pm 2\pi\alpha i}.$$

- Every connected component looks like

- $\chi(G) = 2$
- $\chi'(G) = 2$
- $\chi_c(G)$ is undefined
- $\chi'_c(G)$ is undefined

An example: irrational rotation

Let $\alpha \in (0,1)$ be an irrational number. Consider a subgraph $G_0 = (\mathbb{T} \setminus [0], E)$ of G, where

$$[0] = \{e^{2\pi k\alpha i} \colon k \in \mathbb{Z}\}.$$

- Every connected component still looks like

- $\chi(G_0) = \chi'(G_0) = 2$
- $\mathbb{T} \setminus [0]$ is a Polish space, i.e., a separable completely metrizable space
- $\mathbb{T} \setminus [0]$ is 0-dimensional
- Using Lebesgue density or Baire category, one can show $\chi_c(G_0), \chi_c'(G_0) > 2$
- $\chi_c(G_0) = \chi_c'(G_0) = 3$

Marked groups and Cayley graphs

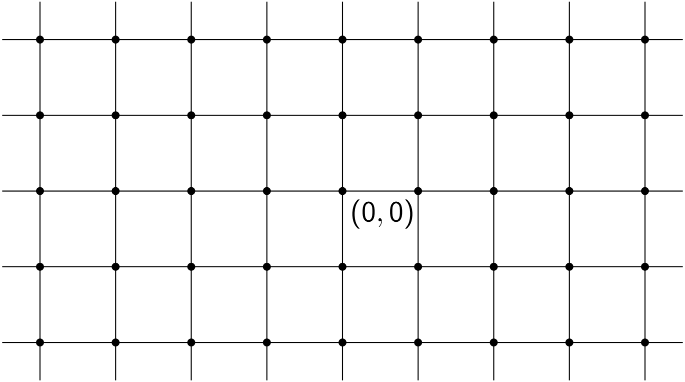
A marked group is a pair (Γ, S) where Γ is a group and S is a finite generating set; typically we require $1_\Gamma \notin S$ and S to be symmetric, i.e., $S = S^{-1}$.

The Cayley graph of (Γ, S) is $G = G(\Gamma, S)$ with

$$V(G) = \Gamma$$
$$E(G) = \{(g, h) \in \Gamma^2 \colon \exists s \in S \; gs = h\}.$$

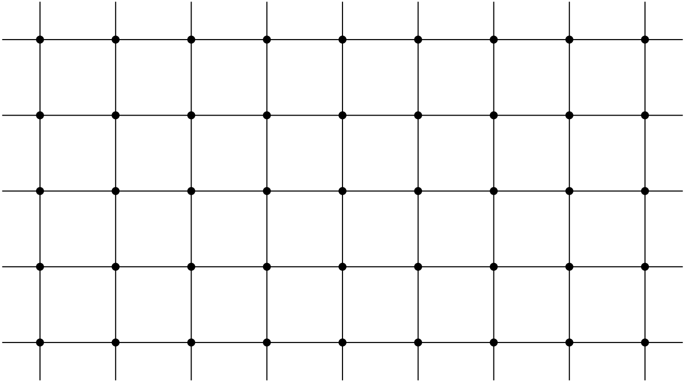
An example

The Cayley graph of the marked group $(\mathbb{Z}^2, \{(\pm 1, 0), (0, \pm 1)\})$:



An example

The Cayley graph of the marked group $(\mathbb{Z}^2, \{(\pm 1, 0), (0, \pm 1)\})$:



Bernoulli shifts and Schreier graphs

For a countable group Γ, the Bernoulli shift action of Γ is the action $\cdot : \Gamma \times 2^\Gamma \to 2^\Gamma$ defined by

$$(g \cdot x)(h) = x(hg).$$

For a marked group (Γ, S), one can define a Schreier graph $G = G(\Gamma, S, 2^\Gamma)$ on $2^\Gamma = V(G)$ by

$$xy \in E(G) \iff \exists s \in S \ (s \cdot x = y).$$

Examples

$2^{\mathbb{Z}^n}$, $F(2^{\mathbb{Z}^2})$, $2^{\mathbb{F}_n}$, $F(2^{\mathbb{F}_n})$

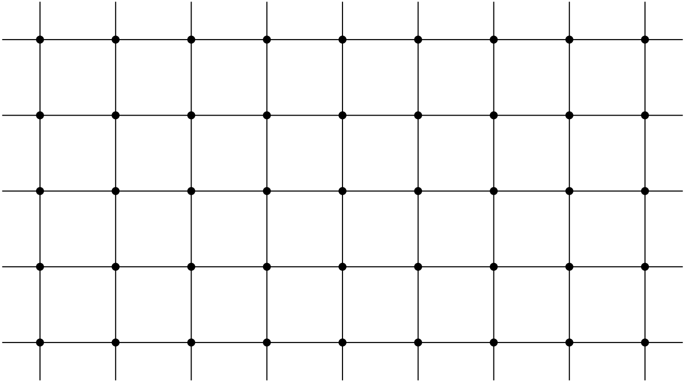
The free part of the Bernoulli shift action of Γ is

$$F(2^\Gamma) = \{x \in 2^\Gamma : \forall g \in \Gamma\ (g \neq 1_\Gamma \longrightarrow g \cdot x \neq x)\}.$$

When there is no danger of confusion, we use $F(2^\Gamma)$ to denote the Schreier graph $G(\Gamma, S, F(2^\Gamma))$.

An example (?)

The Schreier graph on $F(2^{\mathbb{Z}^2})$ consists of continuum many components, with each component a copy of the Cayley graph of $\mathbb{Z}^2$.



The chromatic number

Question What is the chromatic number of the Schreier graph on $F(2^{\mathbb{Z}^2})$?

Answer 1: With AC, the chromatic number of $F(2^{\mathbb{Z}^2})$ is 2

There are *no* Baire measurable or Lebesgue measurable proper 2-colorings on $F(2^{\mathbb{Z}^2})$

Answer 2: (GJKS) The Borel chromatic number of $F(2^{\mathbb{Z}^2})$ is 3

Answer 3: (GJKS) The continuous chromatic number of $F(2^{\mathbb{Z}^2})$ is 4

The edge chromatic number

Question What is the edge chromatic number of the Schreier graph on $F(2^{\mathbb{Z}^2})$?

Answer 1: with AC, the edge chromatic number of $F(2^{\mathbb{Z}^2})$ is 4

Answer 2: (Bencs–Hrušková–Tóth; Chandgotia–Unger; Grebík–Rozhoň; Weilacher) The Borel edge chromatic number of $F(2^{\mathbb{Z}^2})$ is 4

Answer 3: (GJKS) The continuous edge chromatic number of $F(2^{\mathbb{Z}^2})$ is 5

$\mathbb{Z}^2$-subshifts of finite type

Consider $k^{\mathbb{Z}^2}$ for some natural number $k \geq 2$.

- A pattern p is a partial function $p\colon \mathrm{dom}(p) \to k$, where $\mathrm{dom}(p) \subseteq \mathbb{Z}^2$ is finite.
- For $x \in k^{\mathbb{Z}^2}$ and p a pattern, we say that p occurs in x if there is $h \in \mathbb{Z}^2$ such that for all $g \in \mathrm{dom}(p)$, $x(h+g) = p(g)$.
- A $\mathbb{Z}^2$-subshift of finite type is a dynamical system

  $$X_{p_1,\dots,p_n} = \{x \in k^{\mathbb{Z}^2} : p_1, \dots, p_n \text{ do not occur in } x\}$$

  where $p_1, \dots, p_n$ are patterns, with the shift action

  $$(g \cdot x)(h) = x(h+g).$$

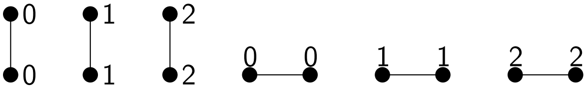
- The patterns $p_1, \dots, p_n$ in the definition of $X_{p_1,\dots,p_n}$ are called forbidden patterns.

$\mathbb{Z}^2$-subshifts of finite type

Example Question

Is there a continuous proper 3-coloring of $F(2^{\mathbb{Z}^2})$?

Consider the $\mathbb{Z}^2$-subshift of finite type $X \subseteq 3^{\mathbb{Z}^2}$, where the forbidden patterns are



Equivalent Question

Is there a continuous equivariant map from $F(2^{\mathbb{Z}^2})$ to X?

$f : F(2^{\mathbb{Z}^2}) \to 3^{\mathbb{Z}^2}$ is equivariant if for all $g \in \mathbb{Z}^2$ and $x \in F(2^{\mathbb{Z}^2})$,

$$f(g \cdot x) = g \cdot f(x).$$

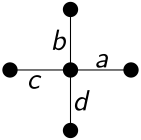
$\mathbb{Z}^2$-subshifts of finite type

Example Question

Is there a continuous proper edge 5-coloring of $F(2^{\mathbb{Z}^2})$?

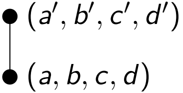
Consider the $\mathbb{Z}^2$-subshift of finite type $Y \subseteq A^{\mathbb{Z}^2}$, where

$$A = \{(a, b, c, d) \colon a, b, c, d \in \{0, 1, 2, 3, 4\} \text{ are distinct}\},$$



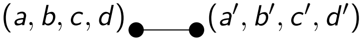
and the forbidden patterns are

$\mathbb{Z}^2$-subshifts of finite type



where $d' \neq b$

and



where $a \neq c'$

Equivalent Question

Is there a continuous equivariant map from $F(2^{\mathbb{Z}^2})$ to Y?

The Subshift Problem

Problem

Given a $\mathbb{Z}^2$-subshift of finite type X, is there a continuous equivariant map from $F(2^{\mathbb{Z}^2})$ to X?

We give a complete (but theoretical) answer to the Subshift Problem for $F(2^{\mathbb{Z}^2})$.

The Twelve Tiles Theorem

Theorem (GJKS)

There are *finite* $\mathbb{Z}^2$-graphs $G_{n,p,q}$, for each triple (n, p, q) of positive integers with $n < p, q$, such that for any $\mathbb{Z}^2$-subshift of finite type X, the following are equivalent:

1. There is a continuous equivariant map from $F(2^{\mathbb{Z}^2})$ to X;
2. There is an equivariant map from $G_{n,p,q}$ to X for some $n < p, q$ with $\gcd(p, q) = 1$;
3. For all n and sufficiently large p, q, there is an equivariant map from $G_{n,p,q}$ to X.

The Twelve Tiles Theorem

Fix $n < p, q$, we define $G_{n,p,q}$.

The definition involves 12 tiles (finite grid graphs):

- 4 torus tiles
- 4 commutativity tiles
- 2 long horizontal tiles
- 2 long vertical tiles

Torus Tiles

$R_\times$	R_c	$R_\times$
R_a		R_a
$R_\times$	R_c	$R_\times$

$G_{ca=ac}$

$R_\times$	R_c	$R_\times$
R_b		R_b
$R_\times$	R_c	$R_\times$

$G_{cb=bc}$

$$R_\times : n \times n,\ \ R_a : n \times (p-n),\ \ R_b : n \times (q-n)$$
$$R_c : (p-n) \times n,\ \ R_d : (q-n) \times n$$

Torus Tiles (continued)

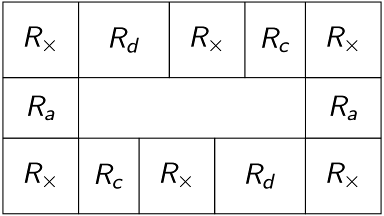
$R_\times$	R_d	$R_\times$
R_a		R_a
$R_\times$	R_d	$R_\times$

$G_{da=ad}$

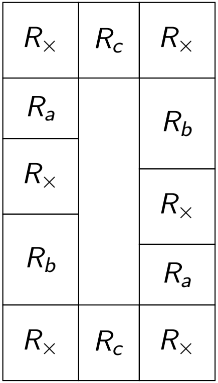
$R_\times$	R_d	$R_\times$
R_b		R_b
$R_\times$	R_d	$R_\times$

$G_{db=bd}$

Commutativity Tiles

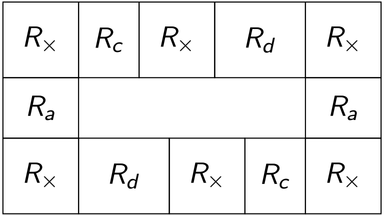


$G_{dca=acd}$

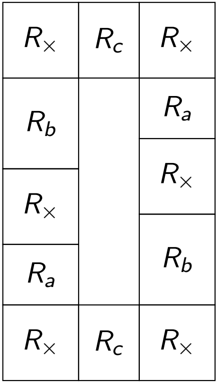


$G_{cba=abc}$

Commutativity Tiles (continued)



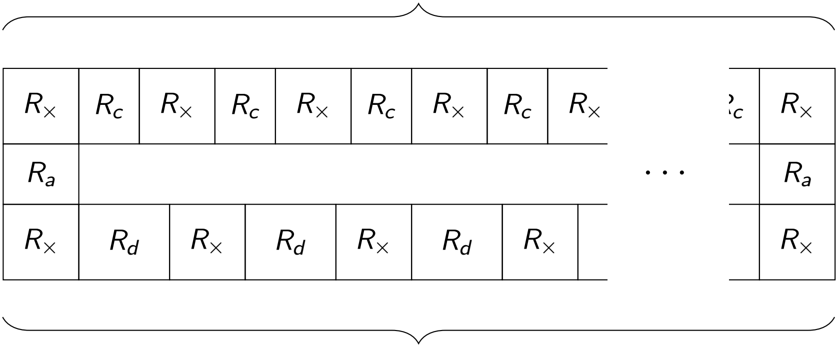
$G_{cda=adc}$



$G_{cab=bac}$

Long Horizontal Tiles

q copies of R_c, $q+1$ copies of $R_\times$

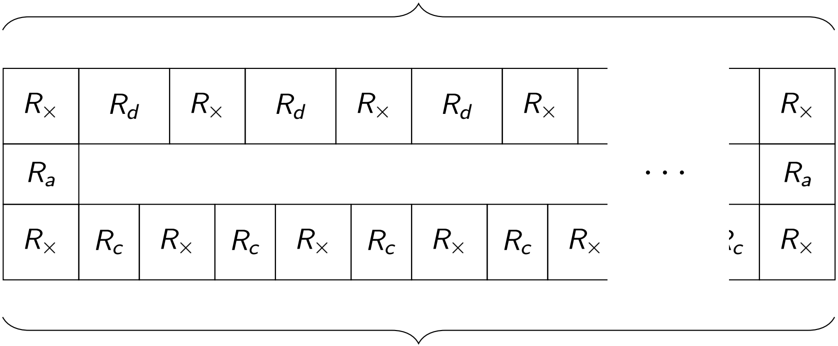


p copies of R_d, $p+1$ copies of $R_\times$

$$G_{c^q a = a d^p}$$

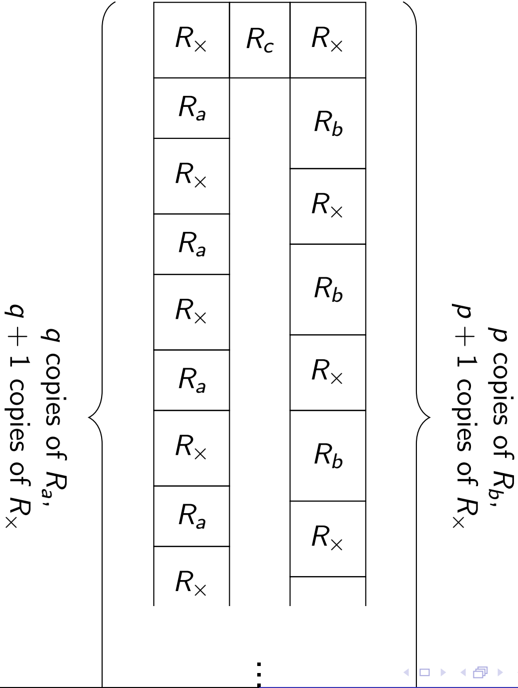
Long Horizontal Tiles (continued)

p copies of R_d, $p+1$ copies of $R_\times$



q copies of R_c, $q+1$ copies of $R_\times$

$G_{d^p a = a c^q}$


$R_\times$
R_c
$R_\times$
R_a
R_b
$R_\times$
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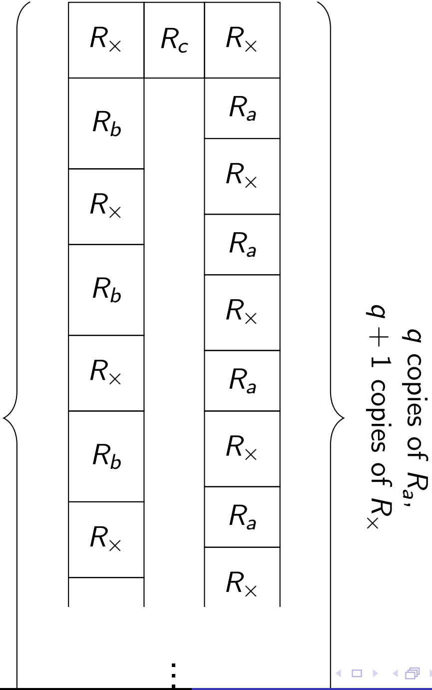
$R_\times$
R_b
$R_\times$
R_a
$R_\times$
q copies of R_a,
$q+1$ copies of $R_\times$
p copies of R_b,
$p+1$ copies of $R_\times$



$R_\times$
R_c
$R_\times$
R_b
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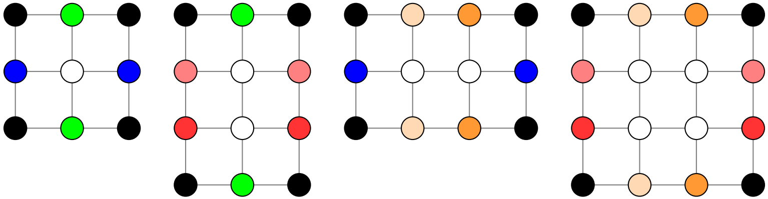
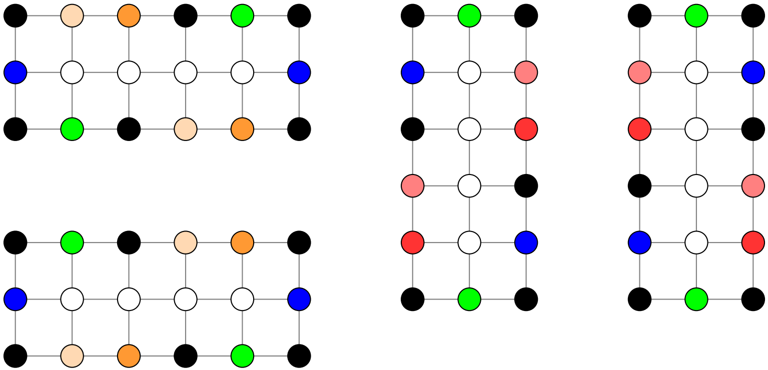
R_a
R_b
$R_\times$
$R_\times$
R_a
$R_\times$
q copies of R_a,
$q+1$ copies of $R_\times$

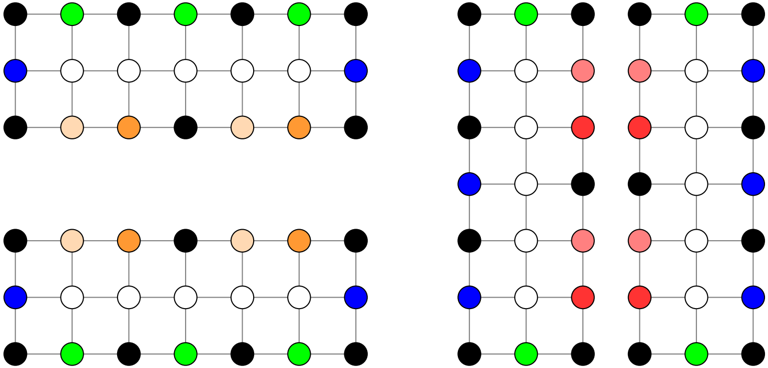
$G_{1,2,3}$: torus tiles



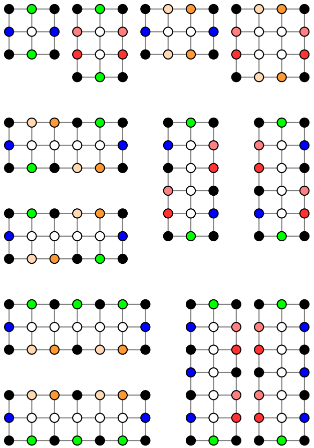
$G_{1,2,3}$: commutativity tiles



$G_{1,2,3}$: long tiles



$G_{1,2,3}$



The Twelve Tiles Theorem

Theorem

For any $\mathbb{Z}^2$-subshift of finite type $X \subseteq k^{\mathbb{Z}^2}$, the following are equivalent:

1. There is a continuous equivariant map from $F(2^{\mathbb{Z}^2})$ to X;
2. For some $n < p, q$ with $\gcd(p, q) = 1$, there is a map $\theta \colon G_{n,p,q} \to k$ which respects X;
3. For all n and sufficiently large p, q, there is a map $\theta \colon G_{n,p,q} \to k$ which respects X.

Undecidability of the Subshift Problem

There are only countably many $\mathbb{Z}^2$-subshifts of finite type, each of which can be coded by a tuple $\langle k; p_1, \ldots, p_n \rangle$.

The Twelve Tiles Theorem implies that the set of all tuples $\langle k; p_1, \ldots, p_n \rangle$ for which there is a continuous equivariant map from $F(2^{\mathbb{Z}^2})$ to $X_{p_1,\ldots,p_n} \subseteq k^{\mathbb{Z}^2}$ is Σ^0_1.

Theorem (GJKS)
The set of all tuples $\langle k; p_1, \ldots, p_n \rangle$ for which is there a continuous equivariant map from $F(2^{\mathbb{Z}^2})$ to $X_{p_1,\ldots,p_n} \subseteq k^{\mathbb{Z}^2}$ is not computable.

There is not a computable bound of how large p and q will be for the first $G_{n,p,q}$ to admit an equivariant map to $X_{p_1,\ldots,p_n}$.

The Graph Homomorphism Problem

Problem

Given a finite graph G, is there a continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ to G?

This is a subproblem of the Subshift Problem. If the Graph Homomorphism Problem is undecidable, so is the Subshift Problem.

Undecidability of the Graph Homomorphism Problem

Theorem (GJKS)
The set of all finite graphs G for which there is a continuous homomorphism from $F(2^{\mathbb{Z}^2})$ to G is undecidable.

We use

Theorem (folklore)
The word problem for finitely presented torsion-free groups is undecidable.

We define a computable reduction of this word problem to the Continuous Graph Homomorphism Problem for $F(2^{\mathbb{Z}^2})$.

Undecidability of the Graph Homomorphism Problem

Start with a finite presentation

$$\mathcal{P}_n = \langle a_1, \ldots, a_k \mid r_1, \ldots, r_l \rangle$$

of a torsion-free group Γ_n, and

$$\text{a distinguished word } w = w(a_1, \ldots, a_k).$$

(*) There is (a lower bound) $\alpha > 0$ such that, if the distinguished word $w \neq e$ in Γ_n, then for all integer $m \geq 1$, w^m is not equal in Γ_n to any word of length $\leq \alpha m$.

Consider

$$\Gamma' = \langle a_1, \ldots, a_k, z | r_1, \ldots, r_l, z^2 w^{-1} \rangle = \langle a_1, \ldots, a_{k+1} \,|\, r_1, \ldots, r_l, r_{l+1} \rangle.$$

Construct a graph G'. G' will have a distinguished vertex v_0. For each of the generators of Γ', we add a sufficiently long cycle β_i of length $\ell_i > 4$ that starts and ends at the vertex v_0. We make the edge sets of these cycles pairwise disjoint. This gives a natural notion of length $\ell(a_i) = \ell_i$ which extends in the obvious manner to reduced words in the free group generated by the a_i. For each word r_j, we wish to add to G' a rectangular grid-graph R_j whose length and width are both > 4 and whose perimeter is equal to $\ell(r_j)$. In order for this to be possible, we will need to make certain that each $\ell(r_j)$ is a large enough even number.

The edges used in the various R_j are pairwise disjoint, and are disjoint from the edges used in the cycles corrresponding to the generators a_i. We then label the edges (say going clockwise, starting with the upper-left vertex) of the boundary of R_j with the edges occurring in the concatenation of the paths corrresponding to the generators in the word r_j.
Finally, G is obtained from G' by forming the quotient graph where vertices on the perimeters of the R_j are identified with the corresponding vertex in one of the a_i.

Undecidability of the Graph Homomorphism Problem

Instead of using the Twelve Tiles Theorem directly, the proof uses some corollaries of the Twelve Tiles Theorem that give positive and negative conditions in terms of the homotopy group of the graph G.

Theorem If there is an odd-length cycle γ which has finite order in $\pi_1^*(G)$, then there is a continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ to G.

Theorem Suppose for every n there are $p, q > n$ with $(p, q) = 1$ such that, for any p-cycle γ in G, γ^q is not a p-th power in $\pi_1^*(G)$. Then there is no continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ to G.

What about $F(2^{\mathbb{Z}})$?

Theorem The Subshift Problem for $F(2^{\mathbb{Z}})$ is decidable.

Question What about $F(2^{\mathbb{Z}^n})$ for $n > 2$?

Question What about $F(2^\Gamma)$ for other groups Γ?

Question What about the Borel Subshift Problem, namely the existence of Borel equivariant maps from $F(2^{\mathbb{Z}^2})$ to a $\mathbb{Z}^2$-subshift of finite type?

Question Is the conjugacy relation between $\mathbb{Z}$-subshifts of finite type computable?

(Berger 1964) The conjugacy relation between $\mathbb{Z}^2$-subshifts of finite type is undecidable.

(Williams 1973) The conjugacy relation between one-sided $\mathbb{Z}$-subshifts of finite type is decidable.

Thanks!