When $A + xA = \mathbb{R}$?

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The First Result

Theorem (Steinhaus)

If $A \subseteq \mathbb{R}$ is a measurable additive group, then either $A = \mathbb{R}$ or A is null.

Proof.

By Lebesgue's density. Actually if $\mu(A) > 0$, then $A - A = \mathbb{R}$. □

Volkmann and Erdös's results

Theorem (Volkmann and Erdoös)

For any $\alpha \in [0,1]$, *there is an additive group* A *so that* $\mathrm{Dim}_{\mathrm{H}}(A) = \alpha$.

Question (Volkmann and Erdös)

Is there a field A *so that* $\mathrm{Dim}_{\mathrm{H}}(A) \in (0,1)$*?*

Analytic Rings

Theorem (Edgar and Miller; Bourgain)

If A is an analytic ring, then either $A = \mathbb{R}$ or $\mathrm{Dim}_{\mathrm{H}}(A) = 0$.

Mauldin's Result

Theorem

Assuming CH, for any $\alpha \in [0,1]$, there is a field F so that $\mathrm{Dim}_{\mathrm{H}}(F) = \alpha$.

Volkmann and Erdös's question is still open.

Some related questions

Volkmann and Erdös's question is still open.

Question

- *Does there exist an ideal of Turing degrees having Hausdorff dimension in* $(0, 1)$ *How about hyperarithmetic degrees?*
- *Does there are two models* $\mathcal{M} \subseteq \mathcal{N}$ *of ZFC so that* $\mathrm{Dim_H}((\mathbb{R})^{\mathcal{M}}) \in (0, 1)$ *in* $\mathcal{N}$*?*

A note on VE's question.

Proposition

Assuming ZF + AD, Volkmann and Erdös's question has a negative answer.

Proof.

If A is a ring with $\mathrm{Dim}_{\mathrm{H}}(A) > 0$, then A has a compact subset B with positive dimension. Then the ring generated by B is analytic and so is $\mathbb{R}$. □

The Motivation

Why a ring is different than a group?

The multiplication operator seems rather complicated.

$A + xA$ for Rings

Proposition (Ye, Y. and Zhao)

If A is a ring so that $A + xA = \mathbb{R}$ for some real x, then $A = \mathbb{R}$.

Proof.

If A is a ring so that $A + xA = \mathbb{R}$ for some real x, then A is a field. By Artin-Schreier, $A = \mathbb{R}$. □

Point to Set Theorem

Definition

Let $\dim_H^x(y) = \underline{\lim}_{n\to\infty} \frac{K^x(y\upharpoonright n)}{n}$.

Theorem (Lutz and Lutz)

For any set A of reals, $\mathrm{Dim_H}(A) = \min_x \max_{r\in A} \dim_H^x(r)$.

Corollary

If f is a Lipschitz function, then for any set A, $\mathrm{Dim_H}(f(A)) \leq \mathrm{Dim_H}(A)$. *Moreover if A is null, then so is $f(A)$.*

$A + pA$ for Groups

Proposition (Ye, Y. and Zhao)

If A is a group, then for any rational p, $\mathrm{Dim}_{\mathrm{H}}(A) = \mathrm{Dim}_{\mathrm{H}}(A + pA)$. Moreover if A is null, then so is $A + pA$.

Proof.

Since A is a group, $A + pA \subseteq \frac{A}{n}$ for some number n. But $x \mapsto \frac{x}{n}$ is a Lipschitz function. □

$A + xA$ for Groups

Theorem (Ye, Y. and Zhao)

There is a Borel group A with $\mathrm{Dim}_{\mathrm{H}}(A) = \frac{1}{2}$ and a real x so that $A + xA = \mathbb{R}$.

Proof.

$y \in \tilde{A}$ if for any n and $m \in [2 \cdot 3^n, 3^{n+1}]$, $y(m) = 0$. $\tilde{A}$ is a Π^0_1 set. Let A be the group generated by $\tilde{A}$. Then A is a Borel group with Hausdorff dimension $\frac{1}{2}$.

$x(m) = 1$ if and only if $m = 3^n$ for some n.

Now given any real $z \in (0, 1)$, let $b \in \tilde{A}$ be a real so that $b(3^n + k) = z(2 \cdot 3^n + k)$ for any $k \in (0, 3^n]$. Then we can find some $a \in \tilde{A}$ so that $a + bx = z$. □

More on $A + xA$

Actually we can make A have Hausdorff dimension arbitrarily $[\frac{1}{2}, 1]$.

Theorem (Barthelemy Le Gac)

If G and H are additive groups so that $G \cap H = \{0\}$ and $G + H = \mathbb{R}$, then either $G = \mathbb{R}$ or $G = \{0\}$.

So $x = \frac{a}{b}$ for some $a, b \in A$.

Question

Is there a Borel group A with $\mathrm{Dim}_{\mathrm{H}}(A) = 0$ so that there is a real x for which $A + xA = \mathbb{R}$?

A Pathological Example (1)

Definition

A real g is weakly-x-generic if for any dense x-r.e. open set U, $x \in U$.

Lemma

If g is weakly-$a \oplus b \oplus x$-generic, then so are $a + g$, $b \cdot g$, and g^n.

A Pathological Example (2)

Lemma

The collection of weakly-generic reals has Hausdorff dimension 0.

A Pathological Example (3)

Theorem (Ye, Y. and Zhao)

Assuming CH. There is a group A so that $A + xA = \mathbb{R}$ if and only if $x \notin \mathbb{Q}$.

Proof.

Fix an enumeration of $\mathbb{R} \times (\mathbb{R} \setminus \mathbb{Q})$. We do it by a transfinite induction:

At stage α: Let g_α be a real weakly generic to all the reals belonging to the idea (in the Turing reduction sense) generated by the reals before and x_α and y_α. Let $h_\alpha = \frac{x_\alpha - g_\alpha}{y_\alpha}$.

Set A to be the group generated by g_α and h_α. □

A Pathological Example (4)

Proof.

Any real $g \in A$ can be read as $\sum_{i=0}^{n}(s_i g_{\alpha_i} + t_i h_{\alpha_i})$, where s_i and t_i are integers. Since y_α is not rational, g must be weakly-generic.

So $\mathrm{Dim}_{\mathrm{H}}(A) = 0$.

By induction, for any irrational y_α and real x_α, $g_\alpha + y_\alpha h_\alpha = x_\alpha$.

If x is rational, then $A + xA \neq \mathbb{R}$ due to $A \neq \mathbb{R}$. □

The Last Question

Question

Can CH in the theorem be removed?

Thanks