Uniform Martin's Conjecture

in the Enumeration Degrees

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Martin's Conjecture

Martin's Conjecture

Assume $ZF + AD + DC$. Then

1. Let $f : 2^\omega \to 2^\omega$ Turing-invariant. If f is not constant* on a cone, then f is increasing on a cone.
2. The non-constant* Turing-invariant functions are pre-well ordered (up to equality on a cone). Moreover, the successor function in the pre-well order is given by the Turing jump.

Where f' is defined by $f'(x) = f(x)'$, for all $x \in 2^\omega$.

Intuition

The only "natural" way to build an incomputable set is with the Turing jump (and iterations of it).

What does "natural" mean?

The key idea behind Martin's conjecture is that natural incomputable sets have two properties:

- They are definable
- Their constructions relativize

These two properties can be expressed mathematically!

Relativization

Any natural way to build an incomputable set X should allow me to build an A-incomputable set X^A.

Moreover, if $A \equiv_T B$, then $X^A \equiv_T X^B$.

From sets to functions

The incomputable set X determines a Turing-invariant function $f_X : 2^\omega \to 2^\omega$ given by

$$f_X(A) = X^A.$$

The Axiom of Determinacy

The Gale-Stewart game with payoff set $\mathcal{A} \subseteq \omega^\omega$, denoted $G(\mathcal{A})$, is the infinite game where two players, I and II, alternate playing natural numbers. Then I wins if and only if the resulting sequence is in $\mathcal{A}$.

I	a_0		a_2		$\dots$		a_{2n}		$\dots$	
II		a_1		a_3		$\dots$		a_{2n+1}		$\dots$

We say that $G(\mathcal{A})$ is determined if some player has a winning strategy.

Axiom of Determinacy (AD)

For every $A \subseteq \omega^\omega$, $G(\mathcal{A})$ is determined.

A false axiom

There is only one problem...

Theorem (Gale and Stewart, 1953)

Under ZFC, the Axiom of Determinacy is false.

However,

Theorem (Martin, 1975)

$$ZF \vDash \text{"If } \mathcal{A} \text{ is Borel, then } G(\mathcal{A}) \text{ is determined"}.$$

Theorem (Martin, Steel and Woodin; 1988-1989)

Assuming the existence of enough large cardinals (infinitely many Woodin cardinals and a measurable cardinal above them),

$$L(\mathbb{R}) \vDash AD$$

Some benefits of Determinacy

Martin's Conjecture is usually stated in terms of $ZF + AD + DC$. This has several benefits:

- Avoids unnatural counterexamples created by the Axiom of Choice.
- It is a flexible hypothesis, because its use often "localizes".
- AD is useful to prove structural properties. For example, under AD every set of reals is Lebesgue-measurable and has the perfect set property.
- Allows us to prove Martin's Cone Theorem.

Martin's Cone Theorem

The cone above x is the set $\nabla_x = \{y \in 2^\omega : x \leq_T y\}$

Theorem (Martin, 1968)

Assume AD. Let $\mathcal{A} \subseteq 2^\omega$ be closed under Turing equivalence. Either $\mathcal{A}$ contains a cone or $2^\omega \setminus \mathcal{A}$ contains a cone.

We can define a countably additive measure in $\mathcal{D}$:

$$\mu(\mathcal{A}) = \begin{cases} 1 & \text{if } \mathcal{A} \text{ contains a cone} \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Martin's Cone Theorem (restated))

μ is a countably-complete ultrafilter.

Theorem (Martin's Cone Theorem (restated again))

If $\mathcal{A} \subseteq \mathcal{D}_T$ is cofinal, then $\mathcal{A}$ contains a cone.

Working on a cone

We need to work on a cone to avoid getting counterexamples to Martin's conjecture by *Frankensteining* functions.

The moral of Martin's cone theorem is: if you glue together countably many Turing-invariant functions, one prevails on a cone.

Definition

Let $f, g : 2^\omega \to 2^\omega$. We say that

- $f \leq_T^\nabla g$ if $f(x) \leq_T g(x)$ for all x on some cone.
- f is *constant on a cone* if there is $y \in 2^\omega$ such that $f(x) \equiv_T y$ for all x on some cone.
- f is *increasing on a cone* if $x \leq_T f(x)$ for all x on some cone.

Martin's Conjecture

Martin's Conjecture

Assume $ZF + AD + DC$. Then

1. Let $f : 2^\omega \to 2^\omega$ Turing-invariant. If f is not constan on a cone, then f is increasing on a cone.
2. The relation $\leq_T^\nabla$ pre-well orders the Turing-invariant functions $\leq_T^\nabla$-above the identity. Moreover, if $\mathrm{rank}_T^\nabla(f) = \alpha$, then $\mathrm{rank}_T^\nabla(f') = \alpha + 1$.

Where f' is defined by $f'(x) = f(x)'$, for all $x \in 2^\omega$.

Partial Results

- Part I and II for uniformly Turing-invariant functions. (Steel, 1982; Slaman and Steel, 1988)
- Part I for regressive functions. (Slaman and Steel, 1988)
- Part I for order-preserving functions. (Lutz and Siskind, 2021)
- Part II for Borel order-preserving functions. Moreover, if f is such a function, there is $\alpha < \omega_1^{CK}$ such that

  $$f(x) = x^\alpha \text{ on a cone}$$

  (Slaman and Steel, 1988)
- The uniform conjecture is morally true for the many-one degrees. (Kihara and Montalbán, 2018)
- The conjecture is false in the arithmetic degrees. (Slaman and Steel, ?)

Uniform Martin's Conjecture

Definition

A function $f : 2^\omega \to 2^\omega$ is uniformly Turing-invariant if there is $u : \omega^2 \to \omega^2$ such that for any $x, y \in 2^\omega$

$$x \equiv_T y \,\text{via}(i,j) \text{ implies that } f(x) \equiv_T f(y) \,\text{via}\, u(i,j)$$

Theorem (Slaman and Steel 1988 ; Steel 1982)

Assume $ZF + AD + DC$. Then

1. Let $f : 2^\omega \to 2^\omega$ uniformly Turing-invariant. If $id \not\leq_T^\nabla f$, then f is constant on a cone.
2. The relation $\leq_T^\nabla$ pre-well-orders the uniformly Turing-invariant functions $\leq_T^\nabla$-above the identity. Moreover, if $\text{rank}_T^\nabla(f) = \alpha$, then $\text{rank}_T^\nabla(f') = \alpha + 1$.

Here f' is defined by $f'(x) = f(x)'$, for all $x \in 2^\omega$

The Case for the Uniformity Assumption

Steel's Conjecture

Under AD, if f is Turing-invariant, then there is an uniformly Turing-invariant function g such that $f \equiv_T^\nabla g$.

Notice that Steel's conjecture implies Martin's conjecture.

Montalbán argues that all the philosophical motivation behind Martin's conjecture also holds for the uniform Martin's conjecture.

A Local Approach

Theorem (Bard, 2020)

Assume $ZF + DC$. Let $x \in 2^\omega$ and $f : \deg_T(x) \to 2^\omega$ be uniformly Turing-invariant. Then, either $x \leq_T f(x)$ or f is constant (literally!).

Theorem (Bard, 2020)

Under $ZF + DC + TD$, the previous theorem implies part I of the uniform Martin's conjecture.

Turing Determinacy (TD) is the statement "every set of Turing degrees either contains a cone, or is disjoint from a cone".

Martin's cone theorem says "$ZF + AD \vDash TD$".

Metamathematics of the uniform Martin's conjecture

Theorem (Bard, 2020)

Under $ZF + DC$, the following are equivalent:

- *Part I of the uniform Martin's conjecture.*
- *Turing Determinacy.*

Theorem (Bard, Chong, Wang, Woodin, Yu)

The following are equivalent over ZFC:

1. *Projective Determinacy.*
2. *Projective Turing Determinacy.*
3. *Part 1 of the projective uniform Martin's conjecture.*
4. *Part 2 of the projective uniform Martin's conjecture.*

Woodin (unpublished) proved $(1) \Leftrightarrow (2)$. Chong, Wang and Yu (2010) proved $(1) \Leftrightarrow (4)$. The proof of the previous theorem by Bard "localizes" to give $(2) \Leftrightarrow (3)$.

Enumeration Reduction

Definition (Friedberg and Rogers, 1959)

Let $A, B \subseteq \omega$. We say $A \leq_e B$ (via i) if there is a program that transforms an enumeration of B into an enumeration of A.

The program is a c.e. table of axioms Γ_i of the form

$$\text{If } \{x_1, \ldots, x_k\} \subseteq B \text{ then } x \in A$$

We say that $A = \Gamma_i(B)$.

Intuition

$A \leq_e B$ means that using positive information about B, we can compute all positive information about A. In contrast, $A \leq_T B$ means that using positive and negative information about B, we can compute positive and negative information about A.

Enumeration Degrees and Turing Degrees

Definition

We say that A is *enumeration equivalent* to B, denoted by $A \equiv_e B$, if $A \leq_e B$ and $B \leq_e A$.

The *Enumeration Degrees* are the following structure:

$$\mathcal{D}_e = (2^\omega / \equiv_e, \leq)$$

Theorem

For any $A, B \in 2^\omega$

$$A \leq_T B \text{ if and only if } A \oplus \overline{A} \leq_e B \oplus \overline{B}$$

This means that the Turing degrees embed into the enumeration degrees via

$$\iota(A) = A \oplus \overline{A}$$

Total and Cototal degrees

Definitions

- A set A is total if $\overline{A} \leq_e A$.
- An enumeration degree is total if it contains a total set.
- A set A is cototal if $A \leq_e \overline{A}$.
- An enumeration degree is cototal if it contains a cototal set.

- Total degrees are exactly the degrees in the range of ι.
- Every total degree is cototal.
- There is a cototal degree that is not total.
- Not every degree is cototal.

Jump and Skip

Definition

- The *enumeration jump* is the map $A \mapsto K^A \oplus \overline{K^A} = A'$
- The *enumeration skip* is the map $A \mapsto \overline{K^A} = A^\diamond$

Theorem (AGKLMSS, 2019)

- $A <_e A^\diamond$ *if and only if* $\deg_e(A)$ *is cototal. Another way to say this,* $\deg_e(A)$ *is cotoal iff* $A' = A^\diamond$.
- *There is some* A *such that* $A = (A^\diamond)^\diamond$

The skip is a uniformly enumeration-invariant function that is neither increasing nor constant on any cone!

Enumeration Cone Theorem?

Theorem (Failure of the cone theorem)

The total degrees, the cototal but non-total degrees, and the non-cototal degrees are all cofinal in $\mathcal{D}_e$ but disjoint.

Local Uniform Martin's Conjeture

Theorem (N. C.)

Let $x \in 2^\omega$. If $f : \deg_e(x) \to 2^\omega$ is uniformly enumeration-invariant and non-constant, then

$$x \leq_e f(x) \quad \text{or} \quad x^\diamond \leq_e f(x).$$

Corollary

Part 1 of Martin's Conjecture holds for Turing-to-enumeration uniformly invariant functions.

Global Uniform Martin's Conjecture

Both parts of the conjecture fail if we try to globalize the local result.

Lemma (N. C.)

You can frankenstein countably-many uniformly enumeration-invariant functions into a single uniformly-invariant function.

Theorem (N. C.)

There is a uniformly enumeration-invariant function that is non-constant, not increasing, and incomparable with the identity and the skip on every cone.

Thank You!